DOCUMENT RESUME

ED 184 829 SE 030 438

AUTHOR	Ayre, H. Glenn; And Others
TITLE	Analytic Geometry, Teachers' Commentary, Part 2. Revised Edition.
INSTITUTION	Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY	National Science Foundation, Washington, D.C.
PUB DATE	64
NOTE	259p.; For related documents, see SE 030 435-440 and ED 143 512-513.
EDRS PRICE	MF01/PC11 Plus Postage.
DESCRIPTORS	Algebra; *Analytic Geometry; Geometric Concepts; Geometry; Mathematics Curriculum; Mathematics Instruction; Secondary Education; *Secondary School Mathematics; *Teaching Guides
IDENTIFIERS	*School Mathematics Study Group

ABSTRACT

This is part two of a three-part teacher's guide to an SMSG textbook designed to be used as a one-semester course for twelfth-grade students. For chapters 6 through 10, this guide includes a teacher's commentary and answers to the exercises. (MK)

****
* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *
****

NATIONAL SCIENCE FOUNDATION
COURSE CONTENT IMPROVEMENT
SECTION
OFFICIAL ARCHIVES
Do Not Remove From Office

ED184829

U S DEPARTMENT OF HEALTH,
EDUCATION & WELFARE
NATIONAL INSTITUTE OF
EDUCATION

THIS DOCUMENT HAS BEEN REPRODUCED EXACTLY AS RECEIVED FROM THE PERSON OR ORGANIZATION ORIGINATING IT. POINTS OF VIEW OR OPINIONS STATED DO NOT NECESSARILY REPRESENT OFFICIAL NATIONAL INSTITUTE OF EDUCATION POSITION OR POLICY

"PERMISSION TO REPRODUCE THIS MATERIAL HAS BEEN GRANTED BY
Mary L. Charles
NSF
TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)."

SE 030 438

ANALYTIC GEOMETRY

Teachers' Commentary

Part 2

(revised edition)

Prepared by:

H. Glenn Ayre, Western Illinois University
William E. Briggs, University of Colorado
Daniel Comiskey, The Taft School, Watertown, Connecticut
John Dyer-Bennet, Carleton College, Northfield, Minnesota
Daniel J. Ewy, Fresno State College
Sandra Forsythe, Cubberley High School, Palo Alto, California
James H. Hood, San Jose High School, San Jose, California
Max Kramer, San Jose State College, San Jose, California
Carol V. McCamman, Coolidge High School, Washington, D. C.
William Wernick, Evander Childs High School, New York, New York

Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

© 1963 and 1964 by The Board of Trustees of the Leland Stanford Junior University
All rights reserved

Printed in the United States of America

PHOTOLITHOPRINTED BY CUSHING - MALLOY, INC.
ANN ARBOR, MICHIGAN, UNITED STATES OF AMERICA

ANALYTIC GEOMETRY

Part 2

Teachers' Commentary

Contents

Chapter

6. CURVE SKETCHING AND LOCUS PROBLEMS 191
7. CONIC SECTIONS 269
8. THE LINE AND THE PLANE IN 3-SPACE 313
9. QUADRIC SURFACES 347
10. GEOMETRIC TRANSFORMATION 425

Teacher's Commentary

Chapter 6

CURVE SKETCHING AND LOCUS PROBLEMS

This chapter explores in detail the relation between a curve and its analytic representation. We present some methods of curve-sketching which are probably new to the students, particularly the addition and multiplication of ordinates and the addition of radii. These relate the graphs of certain types of equations to the more familiar graphs of simpler equations.

We then discuss some geometric properties of the curve and see how they can be deduced from its analytic representations. We see how the choice of coordinate system casts its particular light on our analysis, and explore the advantages and disadvantages of each in a variety of situations. The geometric properties we consider are symmetry, extent, periodicity, intercepts, and asymptotes. The treatment is careful but not exhaustive, and students should be encouraged to see any open questions that we leave, and to try to supply some answers. This is the essence of research, and should be so presented.

We suggest some topics that may be explored as extensions of the content of this chapter: asymptotes which are oblique lines; asymptotes which are other curves; "phase displacements", which may be considered additions of abscissas; properties of families of curves; envelopes of families of curves; self-intersecting curves; extensions to three or more dimensions; applications of this content to physics, particularly to periodic phenomena such as radio broadcasting; the relations among period, frequency, velocity of propagation, and wave length; resonance and interference, both in sound and in light; the Doppler effect in sound and in light; and so on. Students are pleased to recognize the Doppler effect in the changing pitch of an automobile siren as it approaches, passes, and recedes from them. They are also pleased to observe the interference of light as they look through an almost closed space between thumb and forefinger.

The teacher is referred to any recently written text in physics, and particularly to the members of the science department in the school. The topics mentioned are suitable for joint investigation through experimental and theoretical approaches. Both students and teachers can benefit from a systematic investigation in depth of any of the topics mentioned, and the opportunity to check experiment with theory and vice versa.

The sine curve is particularly suited to exhibit such matters as boundedness and periodicity. The polar graph of $r = \sin \theta$ exhibits boundedness in that it is entirely contained in a circle of radius more than $\frac{1}{2}$. The periodicity is shown by the fact that as θ increases without limit, the point P will go endlessly around the circle as shown.

Exercises 6-2(a)

It is to be understood that when we ask for bounds for a graph we want the "best" bounds, that is, the most restrictive. Thus, for 1(a), $y = 2 \sin x$, we certainly have bounds ± 10; "better" bounds are ± 5, but the "best" bounds are ± 2 as indicated below.

1. We use the fact that $0 \leq \sin \theta \leq 1$ and $0 \leq \cos \theta \leq 1$ for any θ.

   (a) $-2 \leq y \leq 2$ for any x.

   (b) $-1 \leq y \leq 1$ for any x.

   (c) $1 \leq y \leq 3$ for any x.

   (d) $-\frac{1}{2} \leq y \leq \frac{1}{2}$ for any x.

   (e) Since $0 \leq \sin \theta \leq 1$ for any θ, we have $0 \leq 2 \sin(3x + \frac{\pi}{2}) \leq 2$, and $-2 \leq y \leq 6$.

   (f) We know $0 \leq |0.6 \sin x| \leq 0.6$, and $0 \leq |0.8 \cos x| \leq 0.8$ therefore we have bounds $0 \leq y \leq 1.4$; but we can do better, since the two terms in the sum, being related, do not reach their maximum (or minimum) values for the same value of x.

   Note that $(.6)^2 + (.8)^2 = 1$ therefore we may take $0.6 = \cos t$ and $0.8 = \sin t$, and write $y = \sin x \cos t + \cos x \sin t$, with t as above. Therefore $y = \sin(x + t)$ and we now have $-1 \leq y \leq 1$. These are the best bounds, and the solution to this exercise.

(g) $y = 2 \text{ xin } x + 3 \cos x = \sqrt{2^2 + 3^2}\left(\frac{2}{\sqrt{2^2 + 3^2}} \sin x + \frac{3}{\sqrt{2^2 + 3^2}} \cos x\right)$

$= \sqrt{2^2 + 3^2}\,(\sin x \cos t + \cos x \sin t)$ where $\cos t = \frac{2}{\sqrt{13}}$,

and so on.

Since $y = \sqrt{13} \sin(x + t)$ we have the solution,

$$-\sqrt{13} \leq y \leq \sqrt{13}$$

(h) $y = a \sin x + b \cos x = \sqrt{a^2 + b^2}\left(\frac{a}{\sqrt{a^2 + b^2}} \sin x + \frac{b}{\sqrt{a^2 + b^2}} \cos x\right)$

$= \sqrt{a^2 + b^2}\,(\sin x \cos t + \cos x \sin t) = \sqrt{a^2 + b^2} \sin(x + t)$;

as in the previous solution. Therefore

$$-\sqrt{a^2 + b^2} \leq y \leq \sqrt{a^2 + b^2}$$

(i) $0 \leq y \leq 1$

(j) $y = \sin^2 x - \cos^2 x = -\cos 2x$. Therefore $-1 \leq y \leq 1$.

2. **Bounds**: $a - |b| \leq y \leq a + |b|$

Period: Since $\sin(cx + d) = \sin(cx + d + 2\pi n)$

$= \sin\left(c(x + \frac{2\pi n}{c}) + d\right)$, there will be no change in y if x is increased by $\frac{2\pi n}{c}$ for integral n . Therefore the period is $\frac{2\pi}{c}$.

6-2(b). Symmetry.

We deal only with point and line symmetry. The content of this section is essential to some important transformations of the plane, which will be dealt with in Chapter 10 and its supplement. Students should be cautioned against replacing the phrase "symmetric with respect to the x-axis", by the non-equivalent, "symmetric with the x-axis". Some authors use "wrt" to replace "with respect to". We usually confine the domain of θ thus: $0 \leq \theta < 2\pi$, since the generalization beyond this domain is usually simple.

Exercises 6-2(b)

1. This question repeats Number 8 of Section 5-6, to whose solution you are referred.

2. An ellipse, or a rectangle which is not square; an equilateral triangle; a square.

3. A circle, a line, the plane, a half-plane.

4. Yes. In review exercise Number 17 at the end of this chapter we ask for the proof of a somewhat stronger statement, that symmetry with respect to both of two perpendicular lines requires symmetry with respect to their intersection.

5. No. Consider the graph of $xy = 1$, or $y = x^3$, or the letter S.

6. We summarize the results by tabulating for parts (a), (b), (c), (d), (e), (h), (i), (j), the answers to these questions: Is the graph symmetric with respect to the x-axis?; the y-axis?; origin?; the line $y = x$?; the line $y = -x$?

   (a) No, yes, no, no, no.
   (b) No, no, yes, no, no.
   (c) No, yes, no, no, no.
   (d) No, no, no, no, no.
   (e) No, no, no, yes, no.
   (f) This equation is equivalent to $(x + y)^2 + 2(x + y) + 1 = 2$ or $(x + y + 1)^2 = 2$, whose graph is the pair of lines $x + y + 1 \pm \sqrt{2} = 0$. These lines are parallel and are symmetric with respect to (1) the line midway between them: $x + y + 1 = 0$; (2) each point of this line, that is, each point $\{(x,y) : x = -t\ , y = t - 1\ , \text{for all } t\}$; and (3) each perpendicular to this line, that is each line of the family $x - y + k = 0$.
   (g) This equation is equivalent to $(x + y + 5)(x + y - 2) = 0$, whose graph is the pair of parallel lines: $x + y + 5 = 0$ and $x + y - 2 = 0$. They are symmetric with respect to (1) the line midway between them, $x + y + \frac{3}{2} = 0$; (2) each point of this line, that is, each point $\{(x,y) : x = -t\ , y = t - \frac{3}{2}\ , \text{for all } t\}$; and (3) each perpendicular to this line, that is, to each line of the family $x - y + k = 0$.
   (h) No, no, no, yes, no.

(i) Yes, no, no, no, no.

(j) If n is even: yes, yes, yes, yes, yes;
if n is odd: no, no, no, yes, no.

For parts (k) - (t). We consider only symmetry with respect to the pole and any line through the pole. We present our answers in this order:
Is the graph symmetric with respect to the pole?
What lines through the pole are axes of symmetry for the graph?

(k) Yes; $\theta = 0$, $\theta = \frac{\pi}{2}$.

(ℓ) No; $\theta = 0$ (since the related polar equation is $r = -\sin^2 \theta$) , $\theta = \frac{\pi}{2}$.

(m) No; $\theta = \frac{\pi}{2}$. (This curve is an ovaloid through the points $(2,0)$, $(1,\frac{\pi}{2}),(2,\pi),(3,\frac{3\pi}{2})$.)

(n) No; $\theta = 0$. (This curve is a parabola.)

(o) No; $\theta = \frac{\pi}{2}$. (This curve is an ellipse. It has symmetry with respect to its center, the point $(\frac{3}{4}, \frac{\pi}{2})$, and the lines along its axes. These lines are most easily represented in rectangular coordinates: $x = 0$, which has already been found, and $y = \frac{3}{4}$. This last result could be found by polar methods but will not be discussed further.)

(p) Yes; $\theta = 0$, $\theta = \frac{\pi}{2}$. (This locus is a pair of parallel lines and has, beside the axes of symmetry already mentioned, any line parallel to the polar axis, that is, any member of the family $r = \frac{a}{\sin \theta}$.

(q) Yes; $\theta = \frac{\pi}{4}$, $\theta = \frac{3\pi}{4}$. (This locus is a double loop in the first and third quadrants, crossing at the pole.)

(r) No; $\theta = \frac{\pi}{6}$, $\theta = \frac{5\pi}{6}$, $\theta = \frac{\pi}{2}$. (This locus is a three-leaved rosette, with loops out to $(2,\frac{\pi}{6})$, $(2,\frac{5\pi}{6})$, $(2, \frac{3\pi}{2})$.)

10

(s) No; $\theta = \frac{\pi}{2}$. (The graph is an ovaloid curve through the points $(3,0)$, $(1,\frac{\pi}{2})$, $(3,\pi)$, $(5,\frac{3\pi}{2})$.)

(t) No; $\theta = \frac{\pi}{2}$. (The graph is an ovaloid curve through the points $(a,0)$, $(a+b,\frac{\pi}{2})$, (a,π), $(a-b,\frac{3\pi}{2})$.)

Challenge Problems

1. Given a point P in space and a plane M which does not contain P. The symmetric image of P with respect to M is the point P' such that M is the perpendicular bisector of $\overline{PP'}$. The question of figure-reversal in a mirror can raise some interesting problems. The fact is that there is a top-bottom reversal, as is seen by the reflection of a mountain in the surface of a lake. We could easily see the top-bottom reversal in our persons if we stood on a mirror, or sat at a mirror-top desk. Our normal position of viewing establishes an unconscious vertical plane of reference, usually the perpendicular bisector of the segment joining our eyes. When we lie on our sides this plane is no longer vertical, and the reversal is now from top to bottom. You may grasp these ideas more clearly if you close one eye to help remove the unconscious vertical plane of reference and then consider various relative positions of the mirror, the eye, and the reflected object.

2. The problem is trivial if L is horizontal or vertical. Assume that it is neither. L is the $\perp$ bisector of $\overline{P_1P_2}$, therefore the midpoint of $\overline{P_1P_2}$ must be on L, therefore $a\left(\frac{x_1 + x_2}{2}\right) + b\left(\frac{y_1 + y_2}{2}\right) + c = 0$, or $ax_1 + ax_2 + by_1 + by_2 + 2c = 0$. $\overleftrightarrow{P_1P_2} \perp L$, therefore $\frac{y_2 - y_1}{x_2 - x_1} = \frac{b}{a}$, thus, $bx_1 - bx_2 - ay_1 + ay_2 = 0$. We solve these two equations for x_2 and y_2, and find

$$P_2 = \left(-\frac{(a^2 - b^2)x_1 + 2ab\,y_1 + 2ac}{a^2 + b^2},\ -\frac{2ab\,x_1 + (b^2 - a^2)y_1 + 2bc}{a^2 + b^2}\right).$$

6-2(c). Extent.

The discussions of the examples in the text are done in sufficient detail to meet the requirements of a text at this level. The ideas of this section are a good foundation for the topic of continuity which is so significant in the calculus. We do not discuss functions whose graphs have serious discontinuities; nor the "pathological" curves of higher mathematics. However, it is salutary for the class to discuss the graph of, say, $y = (-1)^x$ which is totally discontinuous and consists of an infinite number of the points of the lines $y = 1$ and $y = -1$.

In this chapter (Page 214), the term "asymptote" has been used with reference to a line to which the points on a graph approach more and more closely, but which contain no points of the graph. This is always true of the vertical asymptotes, i. e. the y-axis or lines parallel to the y-axis. In the second example (see Figure 6 - 9), we note that the x-axis is crossed by the curve at (0,0) but acts as an asymptote for the points of the graph where $x > 1$ and $x < -3$. In common practice such a line is also referred to as a horizontal asymptote. However, it can be proved that such horizontal asymptote may have only a finite number of points in common with the curve.

It is also important to note that it is possible to have asymptotes which are not lines. For example, the parabola $y = x^2$ acts as an asymptote to the curve $y = x^2 + \frac{1}{x}$. You may like to assign this to your better students after discussing the graph of $y = x + \frac{1}{x}$ on page 220.

6-3. Conditions and Graphs (Rectangular Coordinates).

We have taken a good deal of time and space to show how to sketch certain graphs which are related to familiar graphs. Students soon "catch on" and quickly develop a fine competence in this part of their work, often reporting later that this was the most useful part of the course in later applications.

We suggest again a dynamic approach to graphing. Typical questions are, "How could we change the equation to raise the graph 2 units?" , "What happens to the graph if we reverse these signs?" , etc. As in all exercises, the more, the better, but please do not assign all the exercises of Exercise 6-3. You may, of course, use some of them for test items.

Students are always interested in applications of these ideas that come within their immediate experience. You should point out that the graph of the equation $y = a + b \sin cx$ is a simplified version of broadcast waves that are received by their radio and television sets. An increase in a has the effect of raising the "bias". Roughly this is what is done on the TV set when we increase the "brightness". An increase in b has the effect of increasing the amplitude. On the TV screen the lights would get lighter and the darks would get darker. This is what is done when we increase the "contrast".

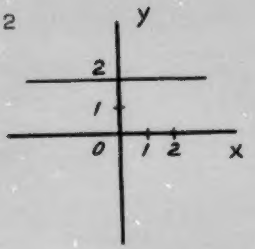
The equation $y = b \sin cx$ also represents roughly the motion of a point on a vibrating string. When we strike a piano key lightly, then heavily we increase the loudness but not the pitch. This situation would correspond in the equation, to increasing b, but keeping c constant.

When we strike two piano keys evenly we have the same loudness but different pitch. This would correspond in the equation to keeping b constant but changing c.

The relationships among mathematics, physics, and music were investigated by the great Greek mathematicians. We leave for individual investigation the extension and development of these ideas to include harmony, resonance, interference, beats, etc., all of which are referred to in any current book on physics.

Exercises 6-3

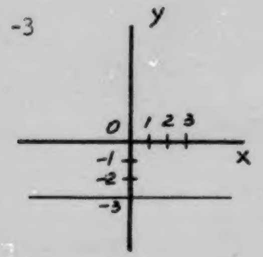
1. $y = 2$



y-intercept: 2

x-intercept: none

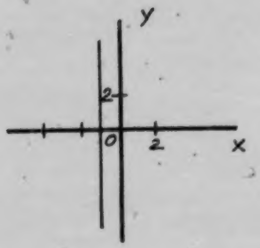
2. $y = -3$



y-intercept: -3

x-intercept: none

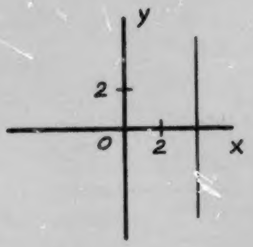
3. $x = -1$



y-intercept: none

x-intercept: -1

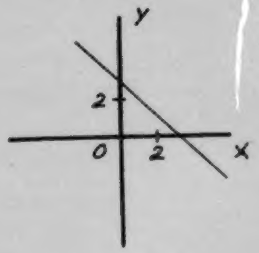
4. $x = 4$



y-intercept: none

x-intercept: 4

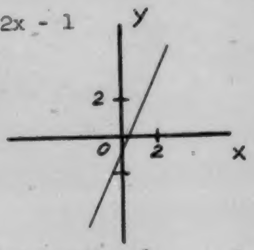
5. $y = -x + 3$



y-intercept: 3

x-intercept: 3

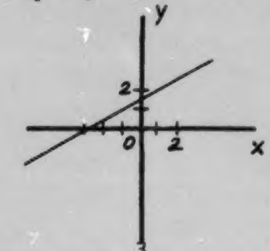
6. $y = 2x - 1$



y-intercept: -1

x-intercept: $\frac{1}{2}$

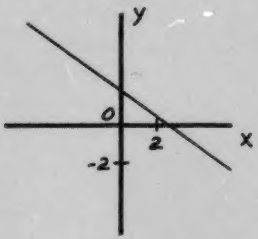
7. $x - 2y + 3 = 0$



y-intercept: $\frac{3}{2}$

x-intercept: -3

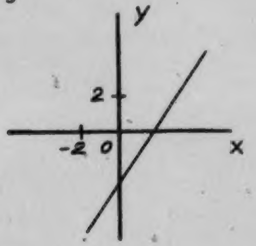
8. $2x + 3y - 5 = 0$



y-intercept: $\frac{5}{3}$

x-intercept: $\frac{5}{2}$

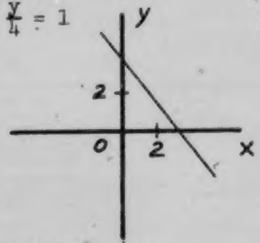
9. $\frac{x}{2} - \frac{y}{3} = 1$



y-intercept: -3

x-intercept: 2

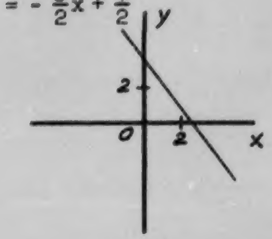
10. $\frac{x}{3} + \frac{y}{4} = 1$



y-intercept: 4

x-intercept: 3

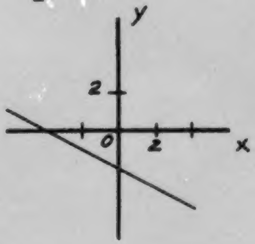
11. $y = -\frac{3}{2}x + \frac{7}{2}$



y-intercept: $\frac{7}{2}$

x-intercept: $\frac{7}{3}$

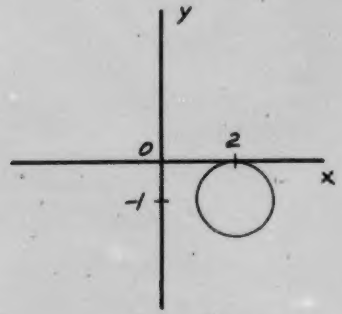
12. $y = -\frac{1}{2}x - 2$



y-intercept: -2

x-intercept: -4

13. $(x - 2)^2 + (y + 1)^2 = 1$



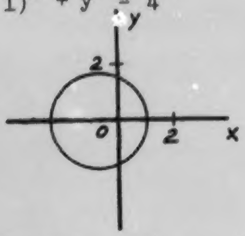
y-intercept: none

x-intercept: 2

center: (2,-1)

bounded

14. $(x + 1)^2 + y^2 = 4$



y-intercept: $\pm\sqrt{3}$

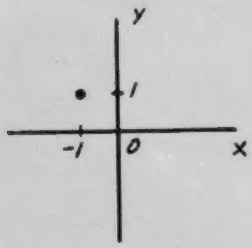
x-intercepts: -3,1

center: (-1,0)

bounded

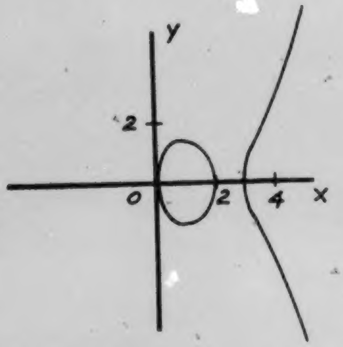
15. $(x + 1)^2 + (y - 1)^2 = 0$

which is $x = -1$ and $y = 1$



The locus is a single point.

16. $y^2 = x(x - 2)(x - 3)$



y-intercept: 0

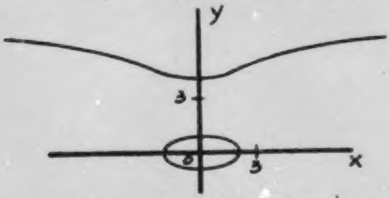
x-intercepts: 0 , 2 , 3

curve is not connected

symmetric in x-axis

no asymptotes

17. $x^2 = (y + 1)(y - 1)(y - 4)$



y-intercepts: -1 , 1 , 4

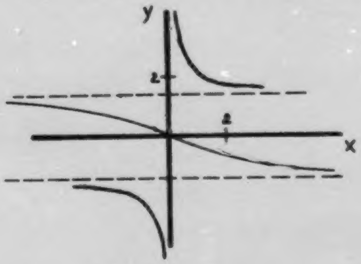
x-intercepts: -2 , 2

curve is not connected

symmetric in y-axis

no asymptotes

18. $xy^2 - 2y - x = 0$



y-intercepts: 0

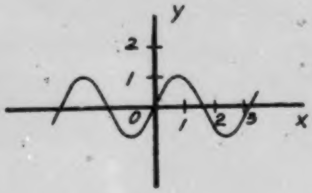
x-intercept: 0

not connected

symmetric in point (0,0)

asymptotes: $y = 1$, $y = -1$, $x = 0$

19. $y = \sin 2x$

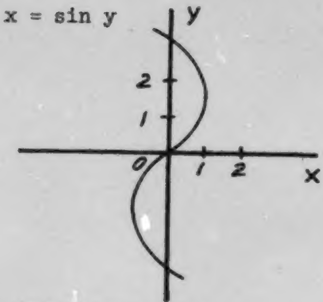


y-intercept: 0

x-intercepts: $n\frac{\pi}{2}$

$-1 \leq y \leq 1$

20. $x = \sin y$

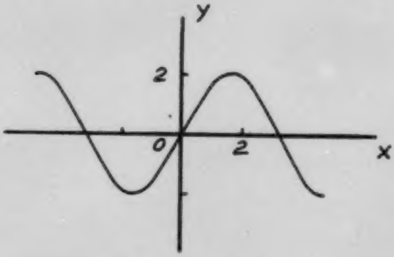


y-intercepts: $n\pi$

x-intercept: 0

$-1 \leq x \leq 1$

21. $y = 2 \sin x$

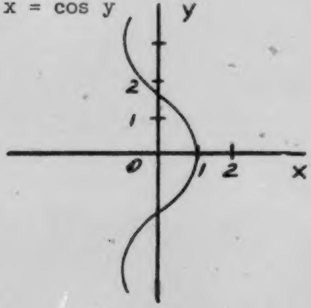


y-intercept: 0

x-intercepts: $n\pi$

$-2 \leq y \leq 2$

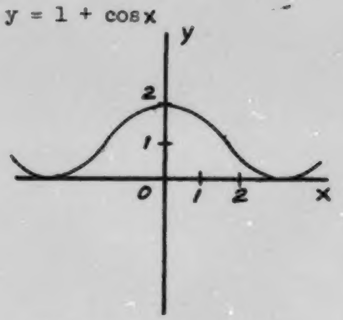
22. $x = \cos y$



y-intercepts: $\frac{\pi}{2} + n\pi$

x-intercept: 1

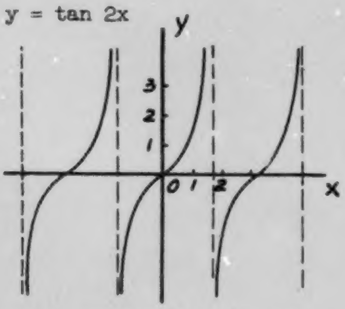
23. $y = 1 + \cos x$



y-intercept: 2

x-intercepts: $(2n + 1)\pi$

24. $y = \tan 2x$

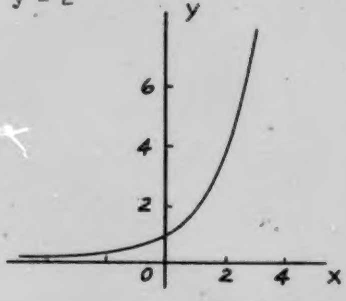


y-intercept: 0

x-intercepts: $n\pi$

asymptotes: $x = \frac{2n + 1}{2}\pi$

25. $y = 2^x$

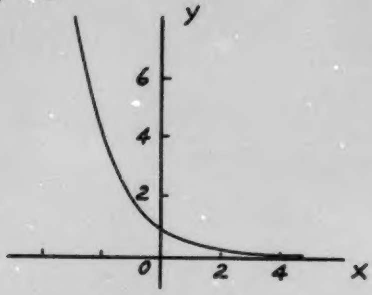


y-intercept: 1

x-intercept: none

asymptote: $y = 0$

26. $y = 2^{-x}$

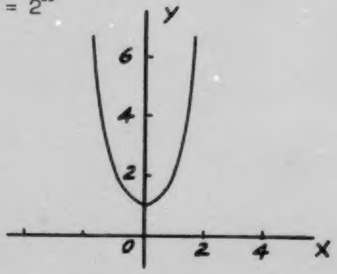


y-intercept: 1

x-intercept: none

asymptote: $y = 0$

27. $y = 2^{x^2}$

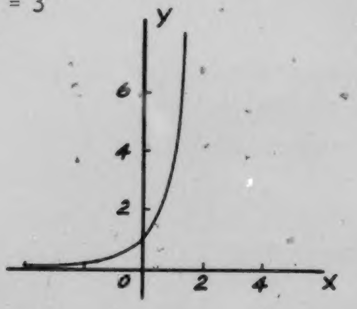


y-intercept: 1

x-intercept: none

symmetric in y-axis

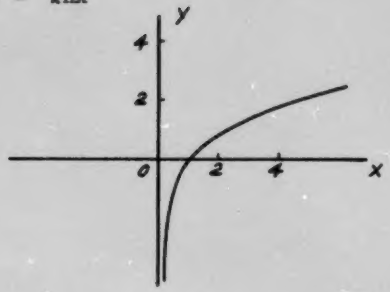
28. $y = 3^{x^3}$



y-intercept: 1

x-intercept: none

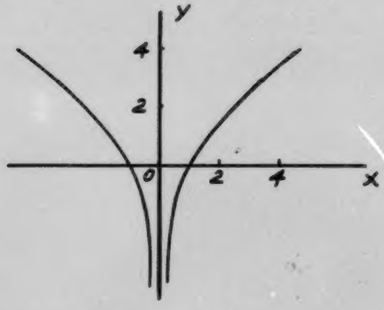
29. $y = \ell n x$



y-intercept: none

x-intercept: 1

30. $y = \ell n x^2$

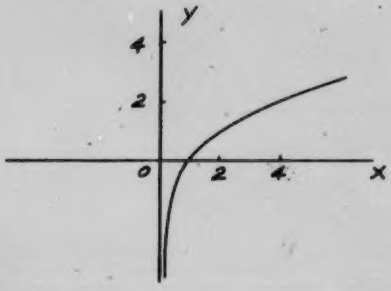


y-intercept: none

x-intercepts: 1 , -1

symmetric wrt y-axis

31. $y = \log_2 x$

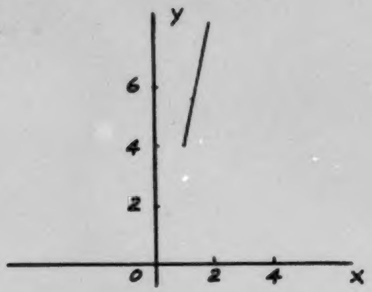


y-intercept: none

x-intercept: 1

asymptote: $x = 0$

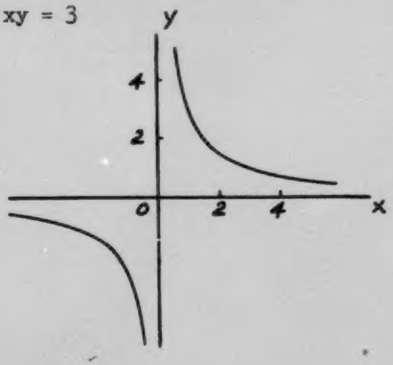
32. $y = 5x - 1$ and $x \geq 1$



no intercepts

no points in left or below $(1,4)$

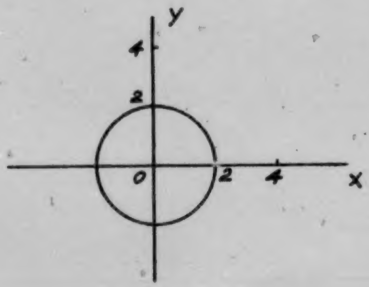
33. $xy = 3$



no intercepts

asymptotes: $x = 0$, $y = 0$

34. $x^2 + y^2 = 2^2$



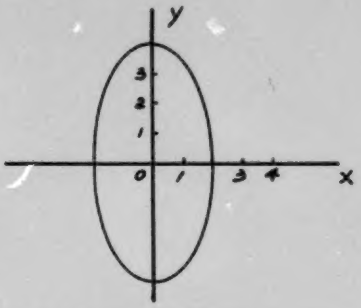
y-intercepts: 2 , -2

x-intercepts: 2 , -2

center: (0,0)

symmetric wrt any line through 0

35. $4x^2 + y^2 = 16$



y-intercepts: 4 , -4

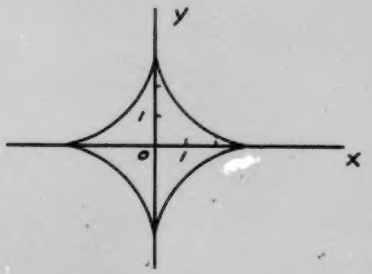
x-intercepts: 2 , -2

symmetric in both axes

$-4 \leq y \leq 4 \quad -2 \leq x \leq 2$

36. $x = 3 \cos^3 \theta \qquad y = 3 \sin^3 \theta$

$x^{2/3} + y^{2/3} = 9^{2/3}$



y-intercepts: 3 , -3

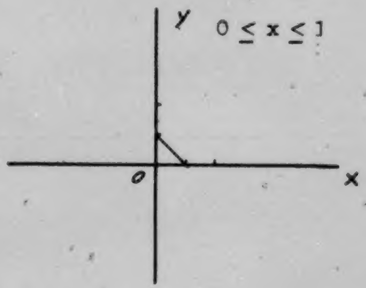
x-intercepts: 3 , -3

symmetric wrt both axes

tangent to axes at corner points

37. $x + y = 1$ $\qquad 0 \leq y \leq 1$

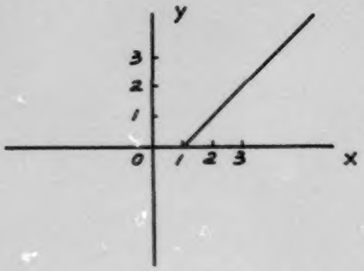
$0 \leq x \leq 1$



y-intercept: 1

x-intercept: 1

38. $y = x - 1$ and $x \geq 1$ and $y \geq 0$

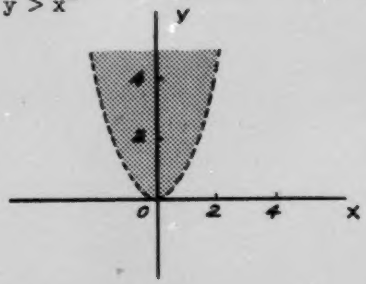


y-intercept: none

x-intercept: 1

restricted to first quadrant

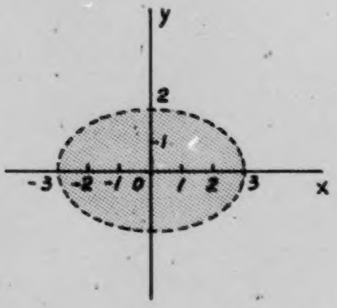
39. $y > x^2$



symmetric wrt y-axis

unbounded

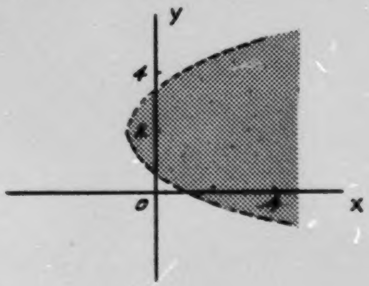
40. $\frac{x^2}{9} + \frac{y^2}{4} < 1$



bounded

symmetric wrt both axes

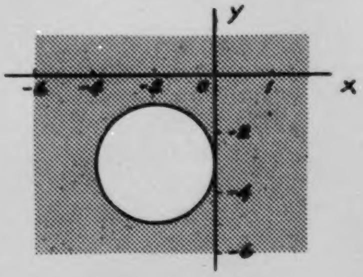
41. $(y - 2)^2 < 2(x + 1)$



unbounded region

symmetric wrt $y = 2$

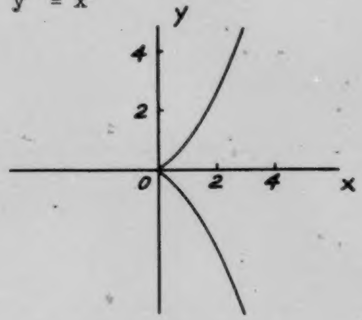
42. $(x + 2)^2 + (y + 3)^2 \geq 4$



unbounded region

center at $(-2,-3)$

43. $y^2 = x^3$

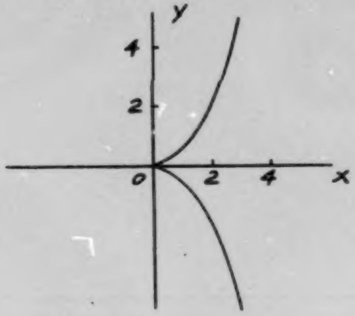


y-intercept: 0

x-intercept: 0

symmetric wrt x-axis

44. $x^3 + xy^2 - 4y^2 = 0$

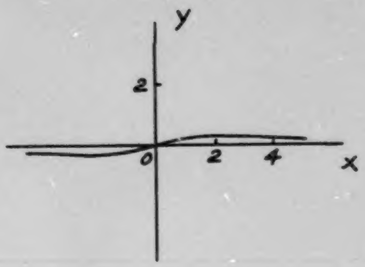


y-intercept: 0

x-intercept: 0

symmetric wrt x-axis

46. $x^2y + 4y - x = 0$



y-intercept: 0

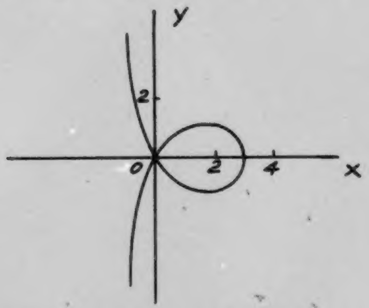
x-intercept: 0

asymptote: $y = 0$

symmetric wrt origin

$\frac{-1}{4} \leq y \leq \frac{1}{4}$

45. $x^3 + xy^2 - 3x^2 + y^2 = 0$

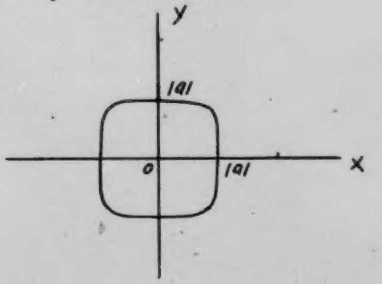


y-intercept: 0

x-intercepts: 0 , 3

asymptote: $x = -1$

47. $x^4 + y^4 = a^4$



y-intercepts: $|a|$, $-|a|$

x-intercepts: $|a|$, $-|a|$

symmetric wrt both axes

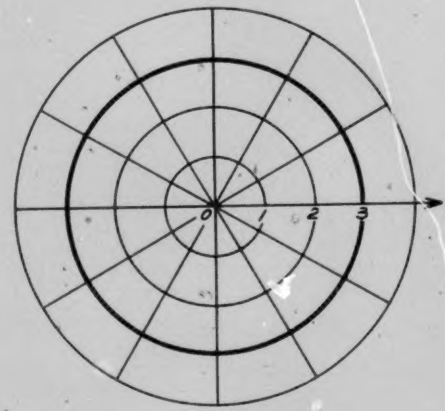
6-4. Graphs and Conditions (Polar Coordinates).

The use of an auxiliary graph in rectangular coordinates, as shown in Example 3 is probably new to the class. It is a useful technique and should be practised in a few exercises until it is understood and becomes a familiar tool. The same may be said for the technique of addition of radii, shown in the same example. We may think of this last technique in a dynamic way, considering the radius, r , as changing, or modulating, as θ changes. Thus, in Figure 6-29(a), as the ray $\overrightarrow{OP}$ rotates counterclockwise the Q-points along that ray also move counterclockwise, but have an extra radial motion, the modulation of the radii. The class might discuss the graphs of $1 + 3 \sin \theta$; $1 + b \sin \theta$; $2 + \sin \theta$; $a + \sin \theta$; and finally $a + b \sin \theta$, for changing values of a and b .

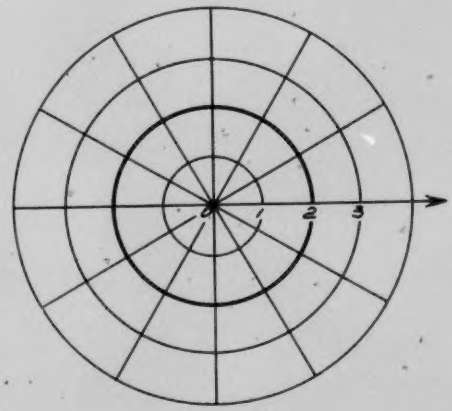
The special ambiguity in the polar coordinates of the pole is an extra ingredient to consider in discussing the intersections of polar graphs. The situation has a geographic analog which students find interesting. If you are at the north pole, which direction is south? The answer is more semantic than factual. If by "south" we mean directly toward the south end of the earth's axis then the answer is: straight down along that axis. If by "south" we mean an available direction of travel along the earth's surface, then the answer is: any direction. If "north" and "south" mean "directly to the ends of the earth's axis", then an object dropped to the surface from a point above the "north pole" will travel simultaneously both north and south!

Exercises 6-4

1. $r = 3$, $r = -3$
   Circle: center 0 , radius 3
   $x^2 + y^2 = 9$



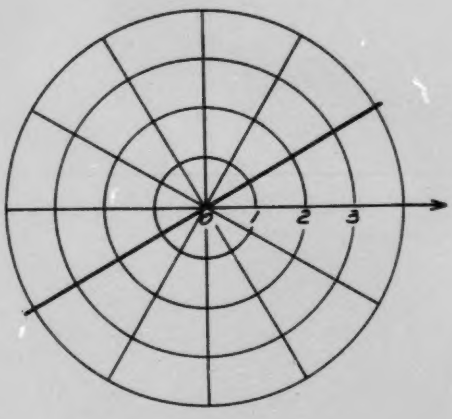
2. $r = -2$, $r = 2$
   Circle: center 0 , radius 2
   $x^2 + y^2 = 4$



3. $\theta = \frac{\pi}{6}$, $\theta = \frac{7\pi}{6}$

Line through 0

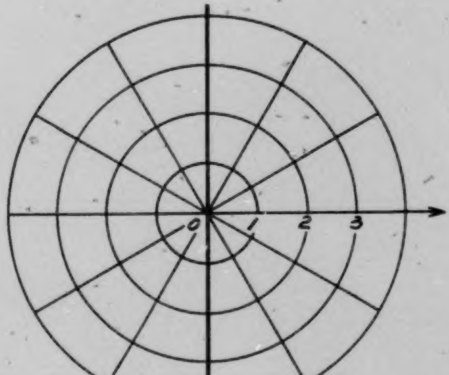
$y = \frac{\sqrt{3}}{3}x$



4. $\theta = -\frac{3\pi}{2}$, $\theta = -\frac{\pi}{2}$

Line through 1

$x = 0$

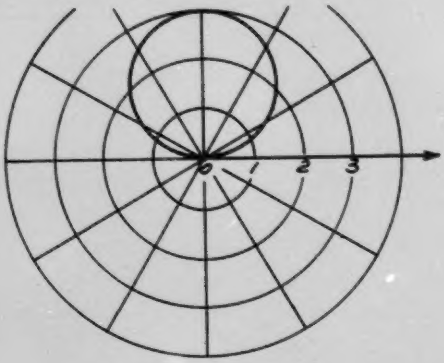


5. $r = 3 \sin \theta$, related equation the same

Circle: center $(\frac{3}{2}, \frac{\pi}{2})$, radius $\frac{3}{2}$

This circle is described twice as the radius vector rotates through 2π .

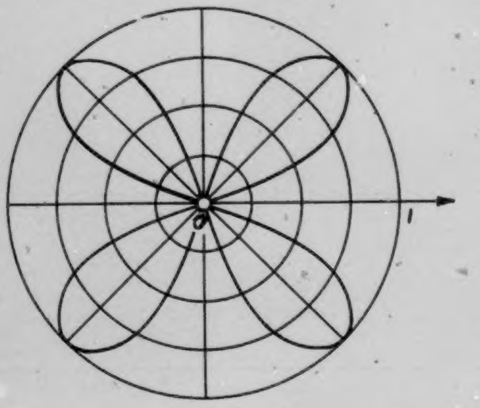
$x^2 + (y - \frac{3}{2})^2 = \frac{9}{4}$



6. $r = \sin 2\theta$, $r = -\sin 2\theta$

The graph is a four-leaved rose.

$(x^2 + y^2)^{3/2} = 2xy$

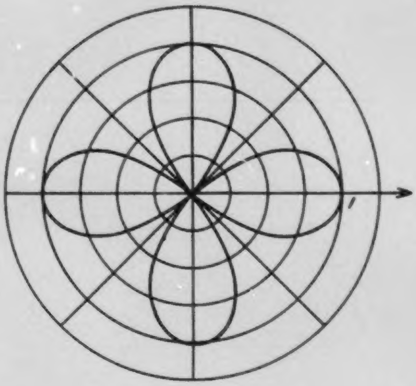


7. $r = \cos 2\theta$, $r = -\cos 2\theta$

Four leafed rose

Symmetric with respect to origin and lines $\theta = 0$, $\theta = \frac{\pi}{4}$, $\theta = \frac{\pi}{2}$

$$(x^2 + y^2)^3 = (x^2 - y^2)^2$$

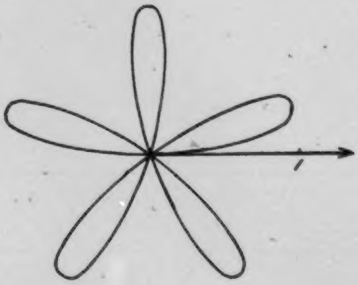


8. $r = \sin 5\theta$, related equation the same

Five leafed rose

Symmetric with respect to origin and lines $\theta = \frac{n\pi}{10}$, $n = 1,3,5,7,9$.

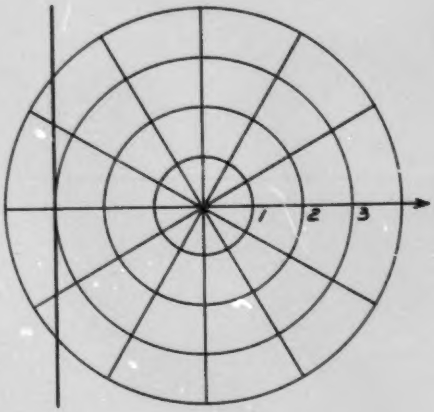
$$(x^2 + y^2)^{3/2} = 5x^4y - 10x^2y^3 + x^5$$



9. $r \cos \theta = -3$, related equation the same

Straight line

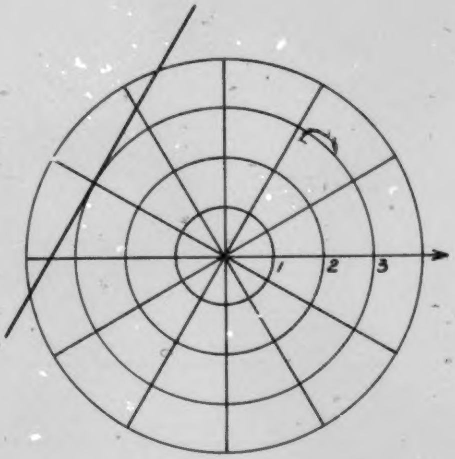
$x = -3$



10. $r \cos(\theta - 150^\circ) = 3$, related equation the same

Straight line

$y = \sqrt{3}x + 6$

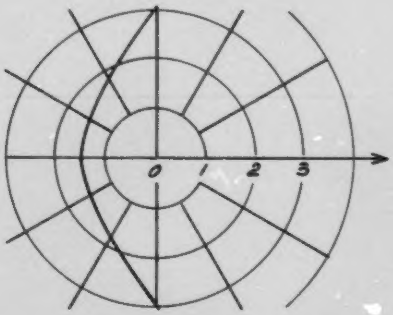


11. $r = \dfrac{3}{1 - \cos\theta}$, $r = \dfrac{-3}{1 + \cos\theta}$

Parabola: focus 0 , directrix $x = -3$

Unbounded. Symmetric with respect to polar axis

$y^2 = 6x + 9$



12. $r = \dfrac{9}{4 - 5\cos\theta}$, $r = \dfrac{-9}{4 + 5\cos\theta}$

Hyperbola with $e = \frac{5}{4}$

foci at 0 and $(10,\pi)$

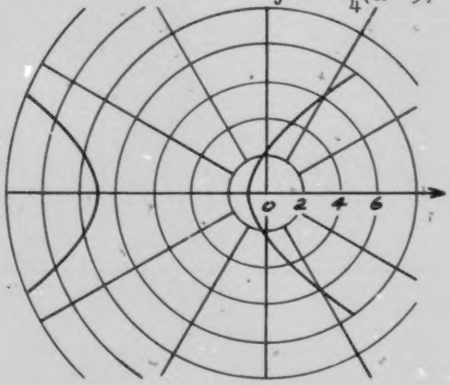
Center at $(5,\pi)$. Unbounded.

Symmetric with respect to center, line $x = -5$, and polar axis.

$9(x + 5)^2 - 16(y)^2 = 144$

The asymptotes are: $y = \frac{3}{4}(x + 5)$

$y = -\frac{3}{4}(x+5)$

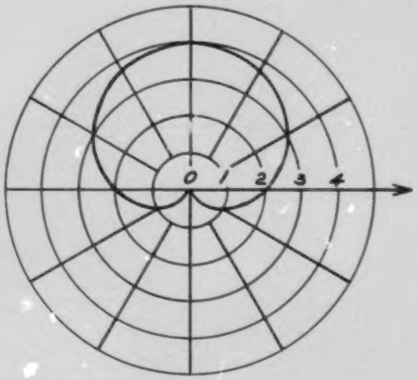


13. $r = 2(1 + \sin\theta)$, $r = 2(\sin\theta - 1)$

Cardiod: Bounded

Symmetric wrt $\theta = 90^\circ$

$(x^2 + y^2)^2 - 4x^2(1 + y) - 4y^3 = 0$



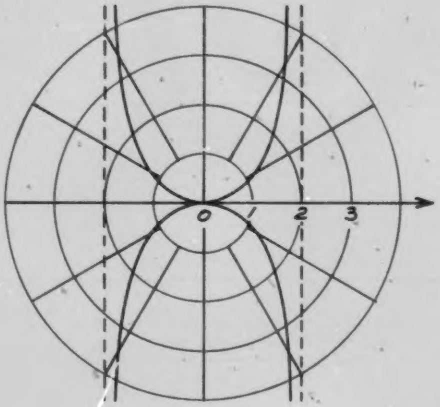
14. $r = 2\tan\theta$, $r = -2\tan\theta$

"Kappa Curve" so called because of its resemblance to the Greek letter kappa, κ

Unbounded. Symmetric wrt origin, $\theta = 0$, $\theta = 90^\circ$.

Vertical asymptotes $x = \pm 2$

$x^4 = y^2(4 - x^2)$

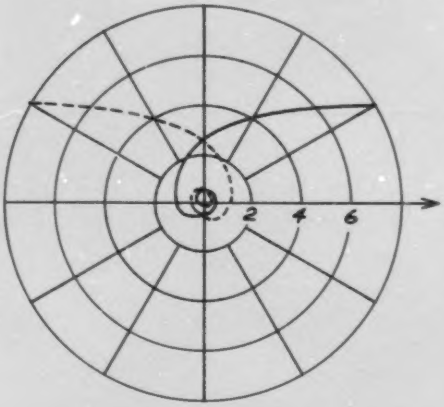


15. $r = \frac{4}{\theta}$, $r = \frac{-4}{\theta + \pi}$

Spiral. Unbounded.

(Solid line corresponds to positive r)

Not defined at $\theta = 0$ or $r = 0$

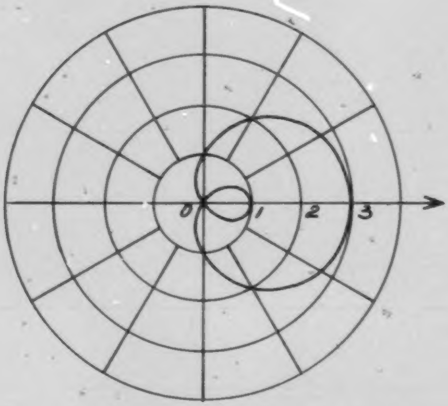


16. $r = 2 \cos \theta - 1$, $r = 2 \cos \theta + 1$

Limacon. Bounded.

Symmetric wrt $\theta = 0$

$(x^2 + y^2)^2 - 4x(x^2 + y^2) + 3x^2 - y^2 = 0$

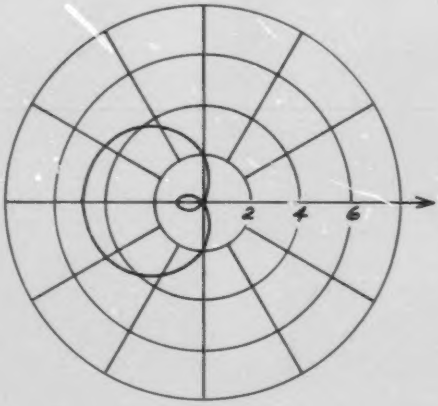


17. $r = 2 - 3 \cos \theta$, $r = -2 - 3 \cos \theta$

Limacon. Bounded.

Symmetric wrt $\theta = 0$

$(x^2 + y^2)^2 + 6x(x^2 + y^2) + 5x^2 - 4y^2 = 0$

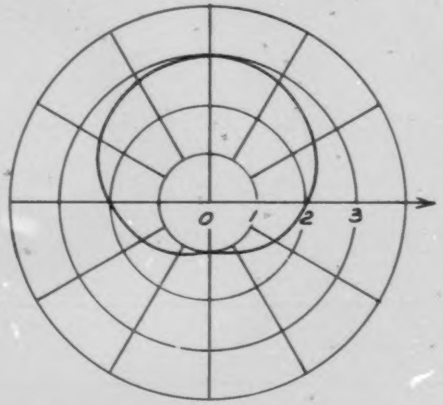


18. $r = 2 + \sin \theta$, $r = \sin \theta - 2$

Cardiod. Bounded.

Symmetric wrt $\theta = 90^\circ$

$(x^2 + y^2)^2 - 2y(x^2 + y^2) - y^2 - 2x^2 = 0$



19. $r^2 = \cos 2\theta$, related equation the same

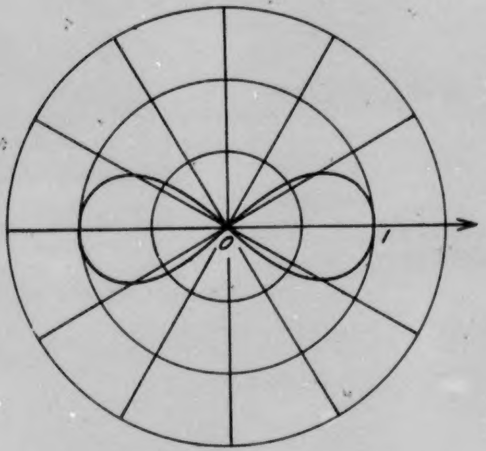
Two leafed rose

Symmetric wrt $\theta = 0$, $\theta = 90^\circ$.

Bounded, restricted to segments.

$-45^\circ \leq \theta \leq 45^\circ$, $135^\circ \leq \theta \leq 225^\circ$

$(x^2 + y^2)^2 = x^2 - y^2$



20. $r^2 = 4 \sin 2\theta$, related equation the same

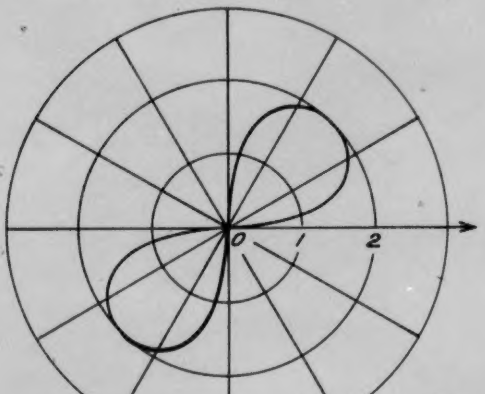
Two leafed rose

Symmetric wrt $\theta = 45^\circ$ and 0

Bounded, restricted to

$0 \leq \theta \leq 90^\circ$, $180^\circ \leq \theta \leq 270^\circ$

$(x^2 + y^2)^2 = 8xy$

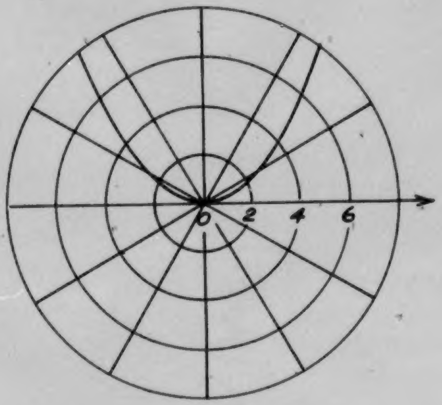


21. $r = 4 \tan \theta \sec \theta$, related equation the same

Parabola. Unbounded.

Symmetric wrt $\theta = 90^\circ$

$y = \frac{1}{4}x^2$

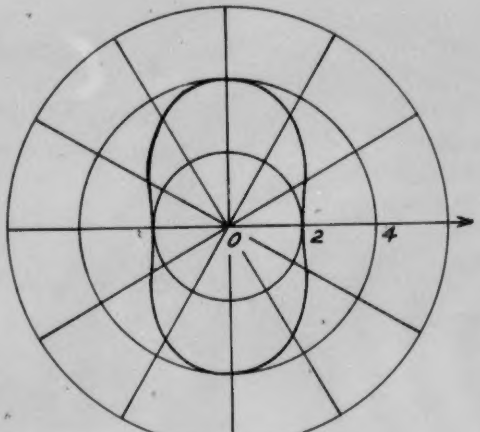


22. $r = 2(1 + \sin^2 \theta)$, $r = -2(1 + \sin^2\theta)$

Bounded.

Symmetric wrt $\theta = 0$, $\theta = 90^\circ$

$(x^2 + y^2)^3 = 4(x^2 + y^2 + y)^2$

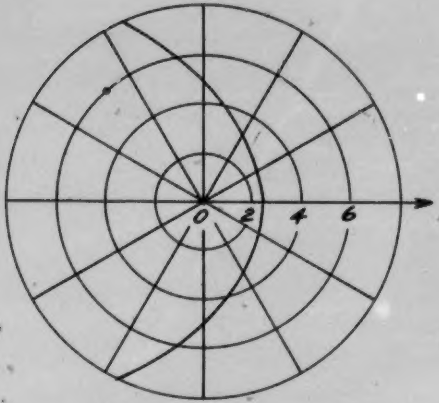


23. $r = \frac{5}{1 + \cos\theta}$, $r = \frac{5}{\cos\theta - 1}$

Parabola. Unbounded.

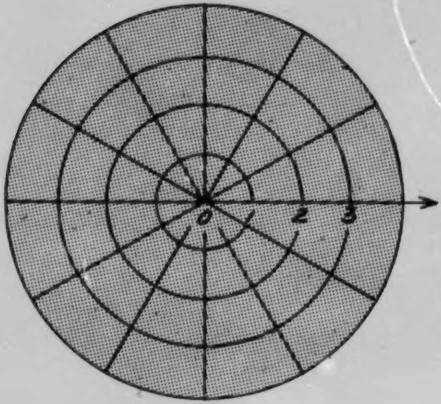
Symmetric wrt $\theta = 0$

$x = -0.1y^2 + 2.5$



24. $r \leq 2$, $r \geq 2$.

The whole plane; every point (r,θ) in the plane may be expressed with negative r.

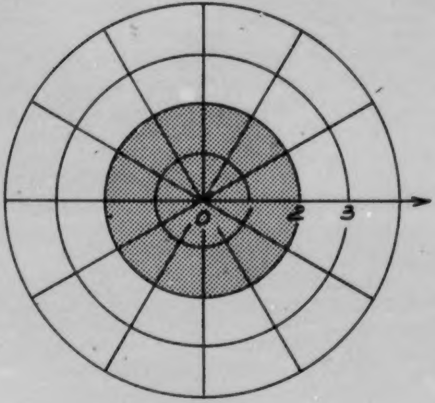


25. $|r| \leq 2$, related equation the same

Disk, boundary included.

Bounded. Symmetric wrt O .

$x^2 + y^2 \leq 4$

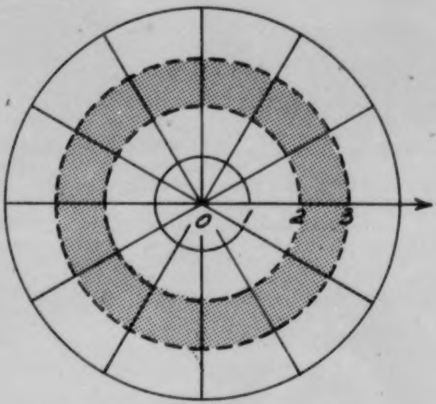


26. $2 < r < 3$, $-3 < r < -2$

Annulus, boundary not included

Bounded. Symmetric wrt O .

$4 < x^2 + y^2 < 9$



28

27. $0 \le \theta \le \frac{\pi}{4}$, $\pi \le \theta \le \frac{5\pi}{4}$

Unbounded. Symmetric

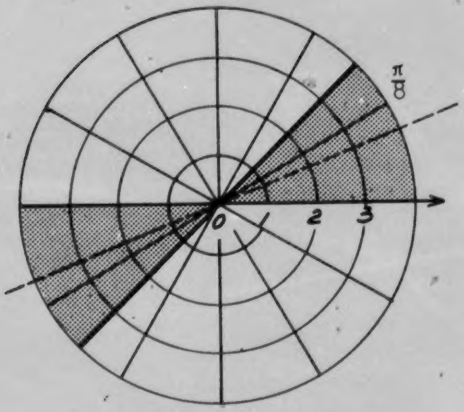
wrt 0 and line $\theta = \frac{\pi}{8}$

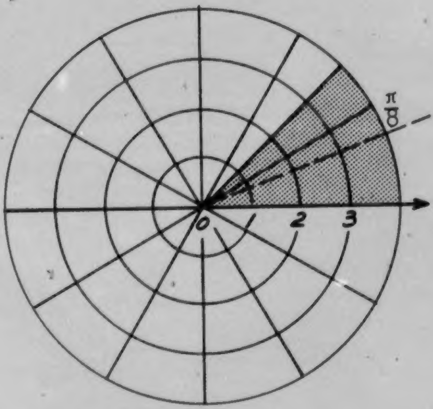
28. $0 \le \theta \le \frac{\pi}{4}$ and $r \ge 0$,

$\pi \le \theta \le \frac{5\pi}{4}$, and $r \le 0$.

Unbounded. Symmetric

wrt line $\theta = \frac{\pi}{8}$





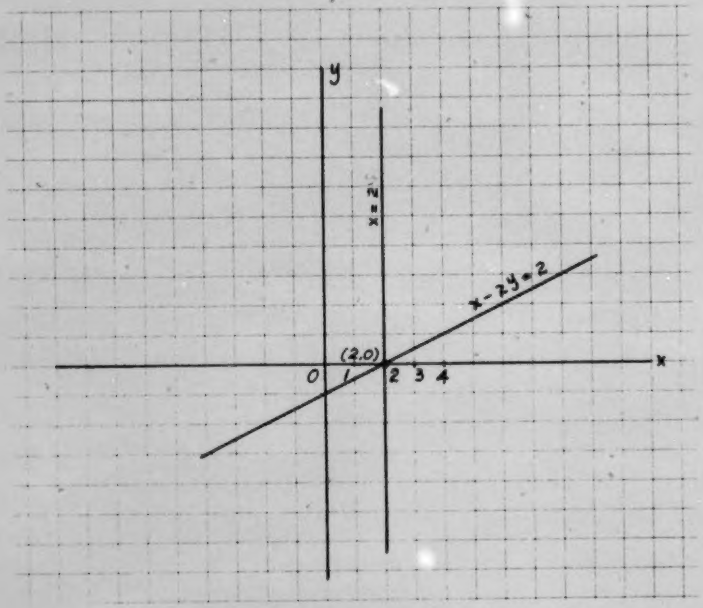
6-5. Intersections of Graphs (Rectangular Coordinates).

This topic has been met in earlier courses and is here treated with a little more generality. The method of linear combinations of functions is used briefly here and more thoroughly in Section 6-7. The exercises are limited to linear and quadratic equations only and present no special difficulties. Higher degree equations have more complicated graphs and present much more difficulties when we consider their intersections.

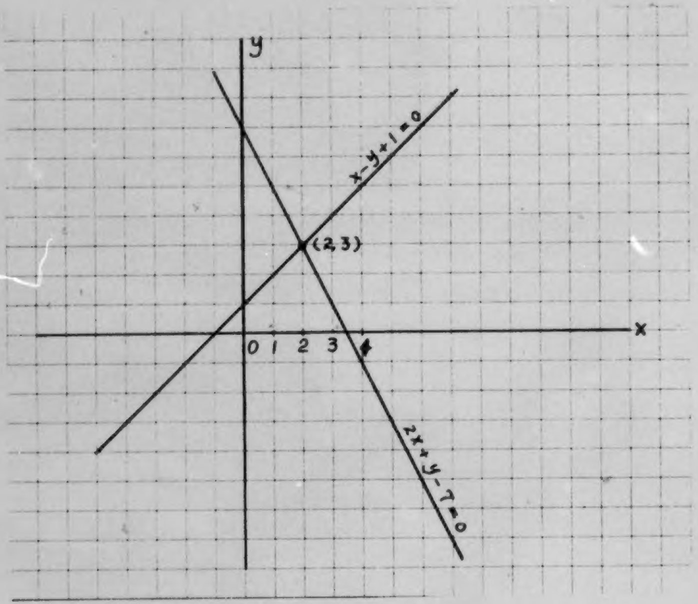
The order, n , of a curve is the maximum number of points of intersection that it may have with a straight line. Students may enjoy discussing the following questions about the orders of curves: What is the relation between the order of a curve and the degree of an equation of it? (Note that we say "an equation", because we have already seen that a curve may have more than one equation.) What is the maximum number of intersections between two curves of orders m and n ? Discuss the order of a closed curve, a self-intersecting curve.

Exercises 6-5

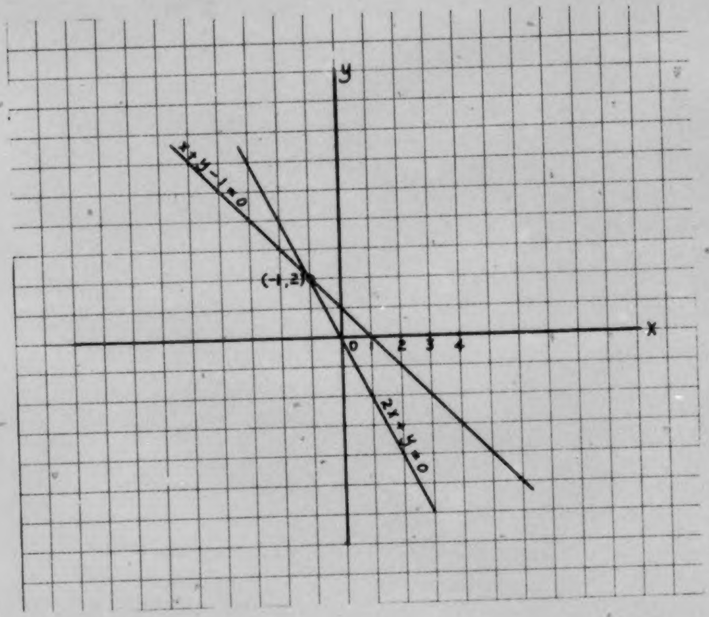
1. The point of intersection is (2,0).



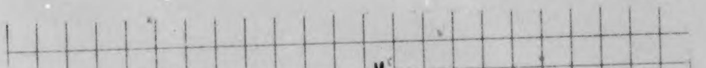
2. The point of intersection is (2,3).



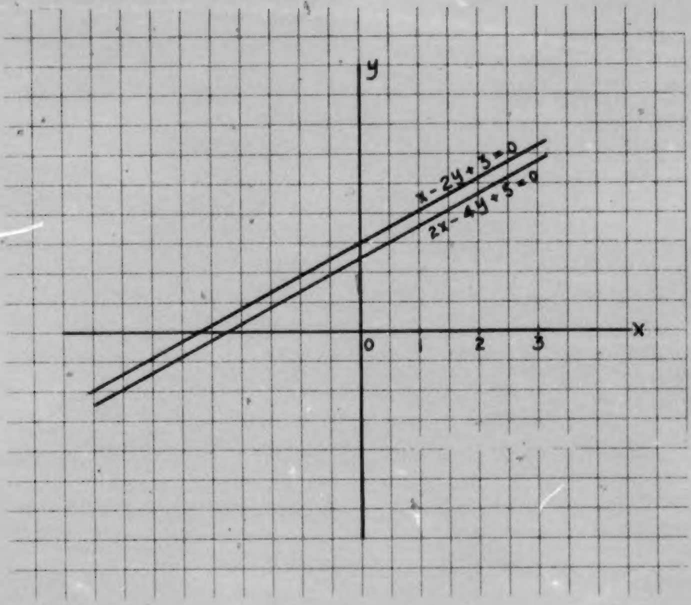
3. The point of intersection is (-1,2).



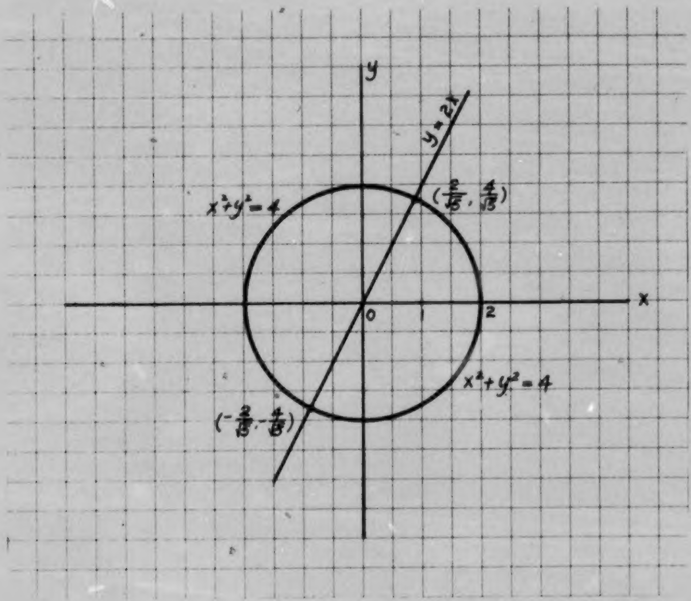
4. The point of intersection is $(\frac{1}{5}, \frac{8}{5})$.



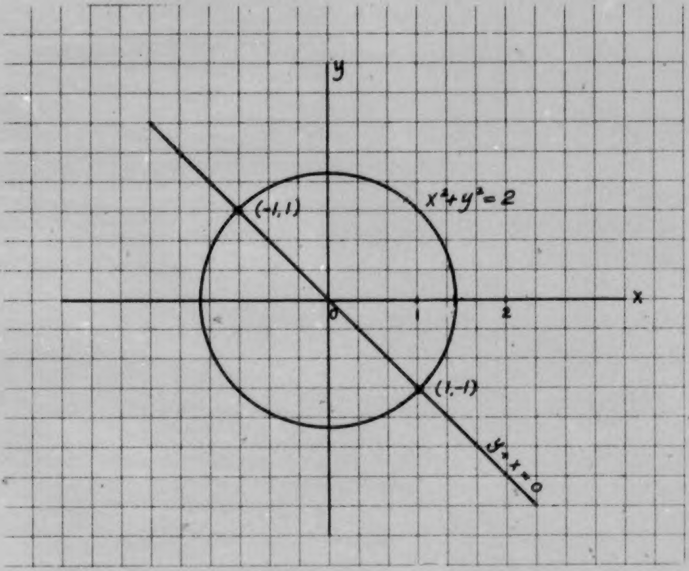
5. The intersection is the null set.



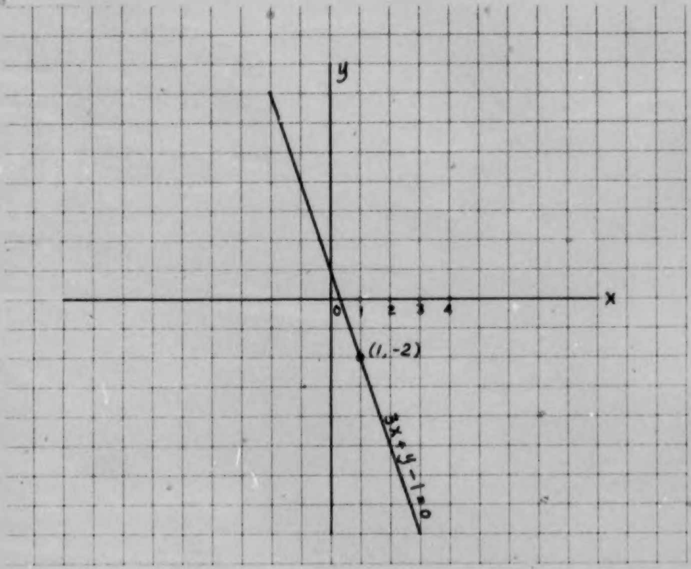
6. The points of intersection are $\left(\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right)$ and $\left(-\frac{2}{\sqrt{5}}, -\frac{4}{\sqrt{5}}\right)$



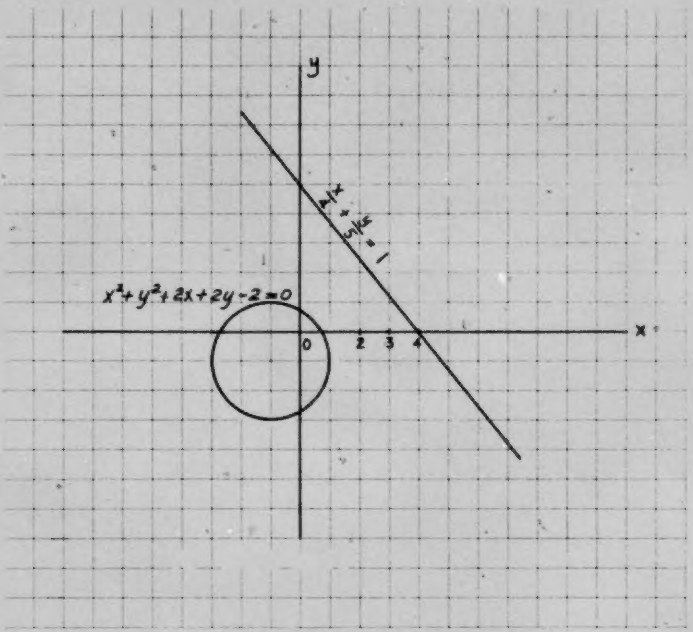
7. The points of intersection are (1, -1) and (-1, 1).


y
x
0
1
2
(-1,1)
(1,-1)
$x^2+y^2=2$
$y+x=0$

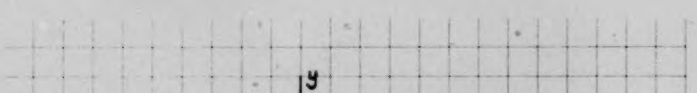
8. The point of intersection is (1, -2).


y
x
0
1
2
3
4
(1,-2)
$3x+y-1=0$

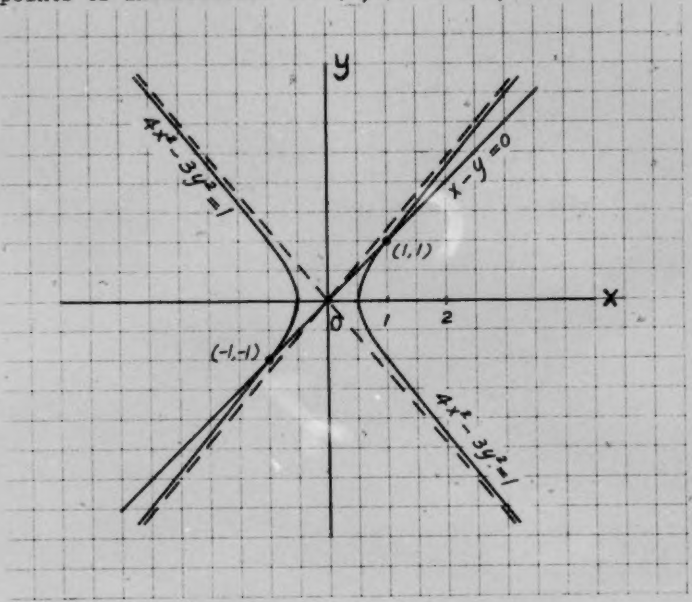
9. The intersection is the null set.



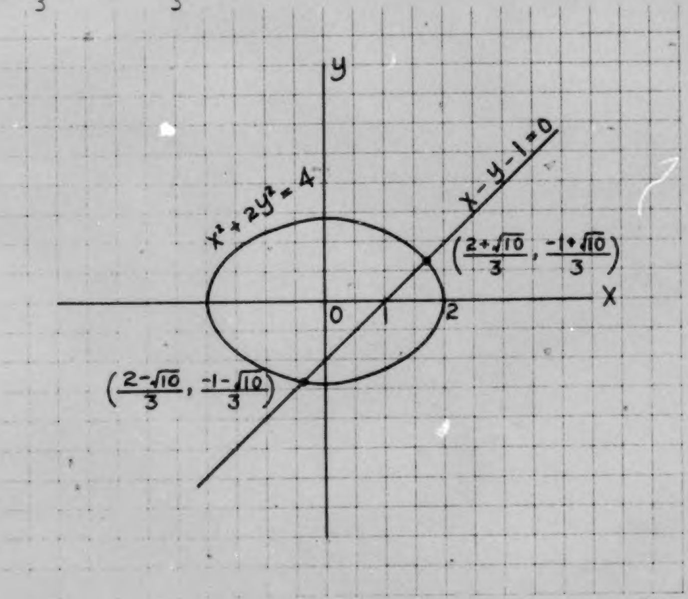
10. The points of intersection are (1,2) and (9,6).



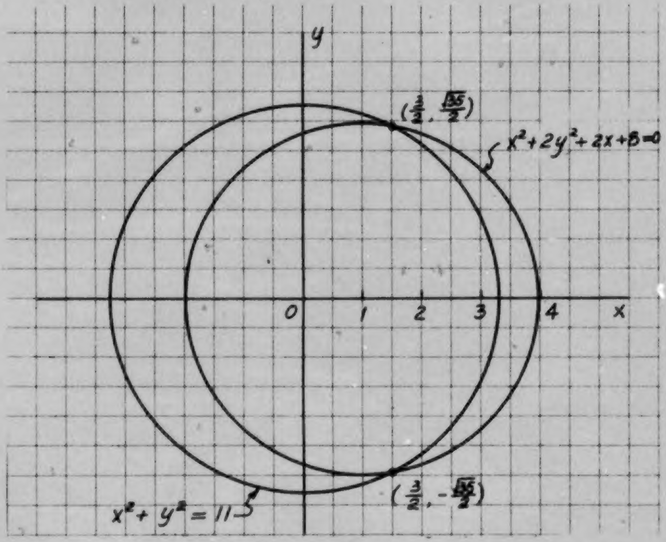
11. The points of intersection are (1,1) and (-1, -1).


y
x
$4x^2 - 3y^2 = 1$
$x - y = 0$
(1,1)
(-1,-1)
0
1
2
$4x^2 - 3y^2 = 1$

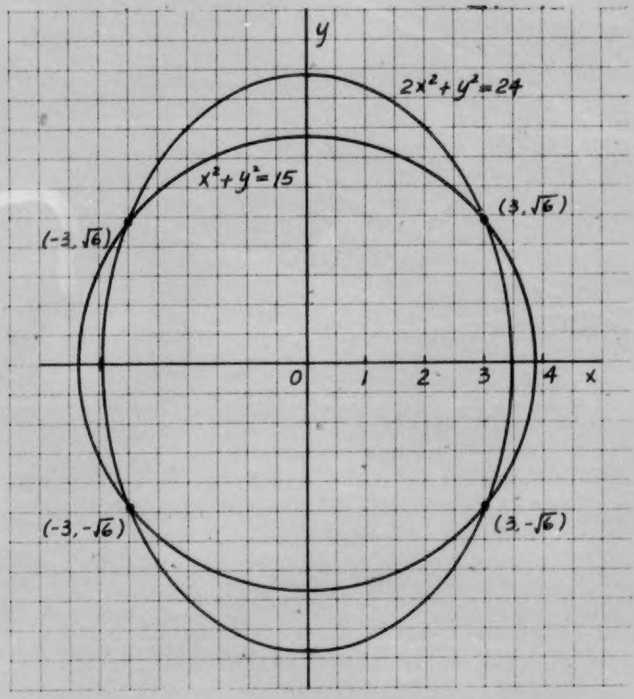
12. The points of intersection are $\left(\frac{2 + \sqrt{10}}{3}, \frac{-1 + \sqrt{10}}{3}\right)$ and $\left(\frac{2 - \sqrt{10}}{3}, \frac{-1 - \sqrt{10}}{3}\right)$


y
x
$x^2 + 2y^2 = 4$
$x - y - 1 = 0$
$\left(\frac{2+\sqrt{10}}{3}, \frac{-1+\sqrt{10}}{3}\right)$
0
1
2
$\left(\frac{2-\sqrt{10}}{3}, \frac{-1-\sqrt{10}}{3}\right)$

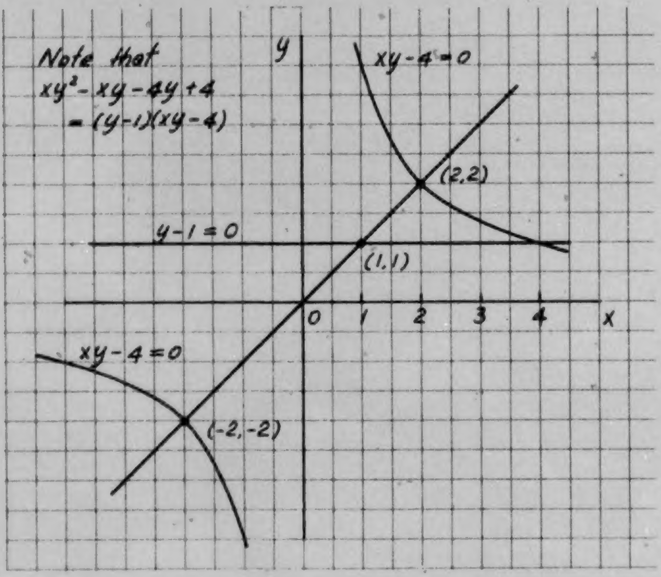
13. The points of intersection are $(\frac{3}{2}, \frac{\sqrt{35}}{2})$ and $(\frac{3}{2}, \frac{-\sqrt{35}}{2})$.



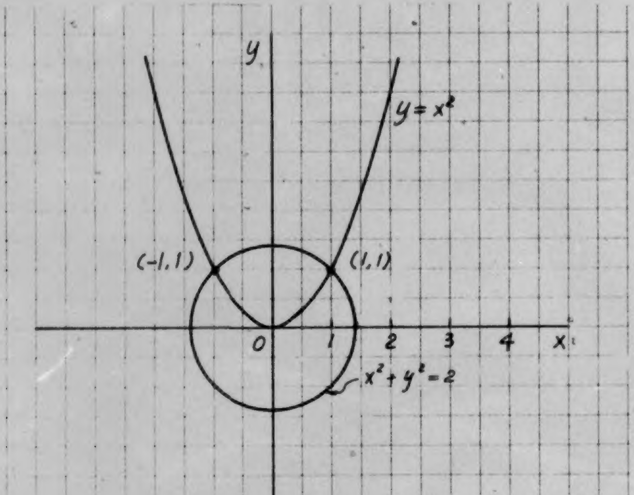
14. The points of intersection are $(3, \sqrt{6})$, $(3, -\sqrt{6})$, $(-3, \sqrt{6})$, and $(-3, -\sqrt{6})$.



15. The points of intersection are (2,2) , (1,1) and (-2,-2).

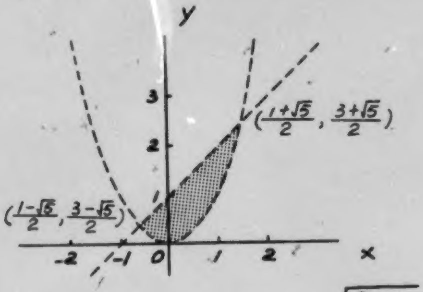


16. The points of intersection are (1,1) and (-1,1).

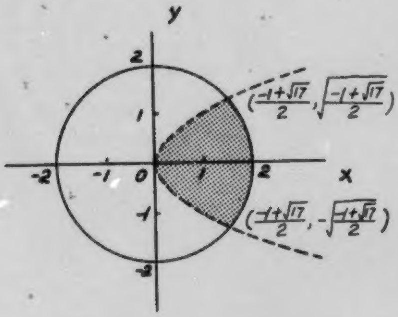


For Problems 17 to 19, the intersection is the shaded region.

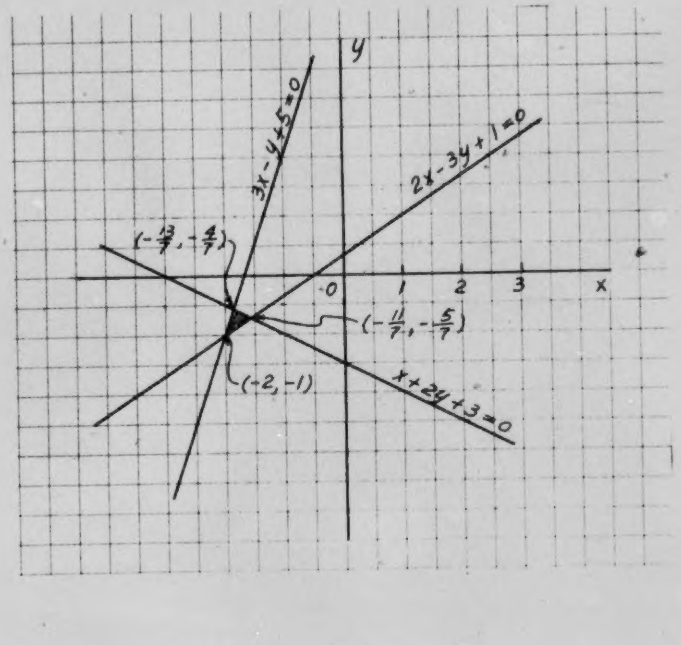
17. $\{(x,y) : y > x^2 \text{ and } y < x + 1\}$


y
3
2
1
$(\frac{1+\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2})$
$(\frac{1-\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2})$
-2
-1
0
1
2
x

18. $\{(x,y) : x > y^2 \text{ and } x \leq \sqrt{y^2 - 4}$


y
2
1
$(\frac{-1+\sqrt{17}}{2}, \sqrt{\frac{-1+\sqrt{17}}{2}})$
-2
-1
0
1
2
x
-1
$(\frac{-1+\sqrt{17}}{2}, -\sqrt{\frac{-1+\sqrt{17}}{2}})$
-2

19.


y
$3x-y+5=0$
$2x-3y+1=0$
$(-\frac{13}{7}, -\frac{4}{7})$
0
1
2
3
x
$(-\frac{11}{7}, -\frac{5}{7})$
$(-2,-1)$
$x+2y+3=0$

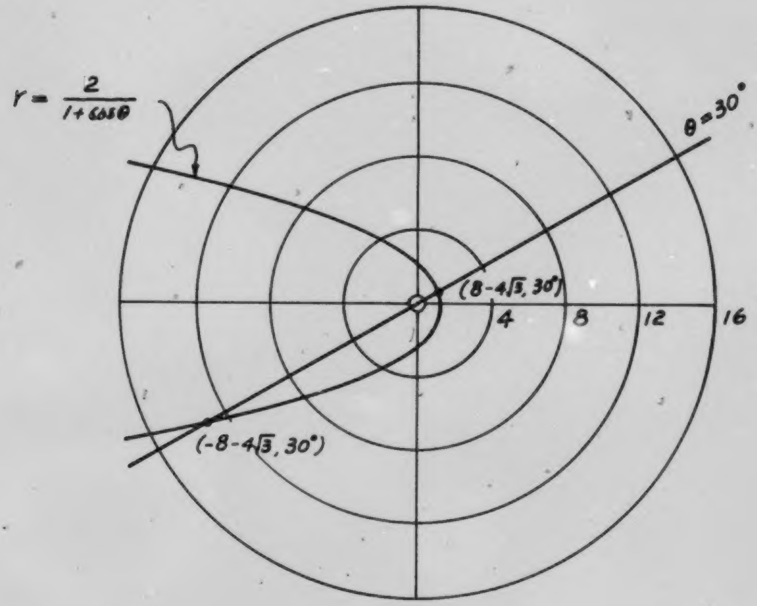
6-6. Intersection of Loci (Polar Coordinates).

Any adequate treatment of this topic must give careful consideration to the special situation at the pole; and to the multiple representations of polar graphs. We have done this in the text, and found the concept of related polar equations particularly useful in finding all intersection points.

Intersections of the graphs of polar inequalities are not treated here because they lead into content beyond the level of this book.

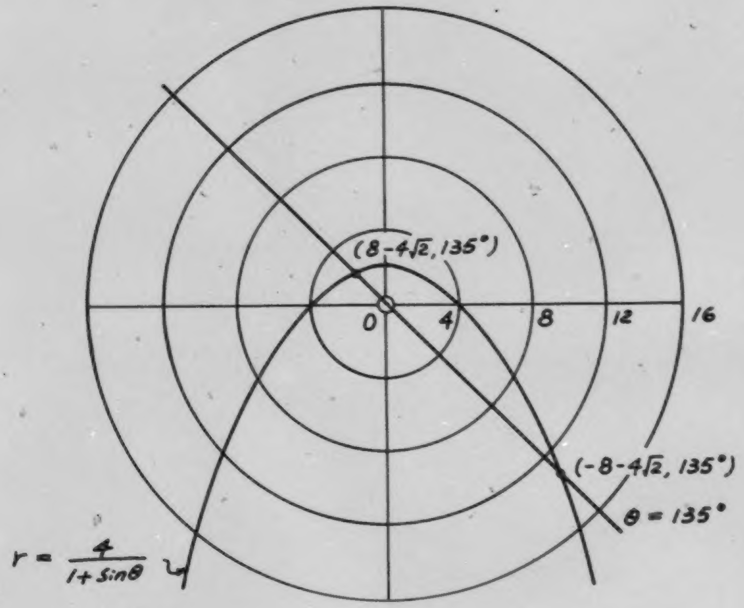
Exercises 6-6

1.



Related equations are $r = -\frac{2}{1 - \cos\theta}$ and $\theta = 210^\circ$.

2.

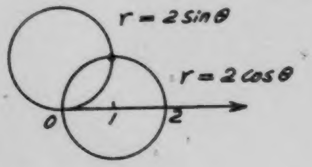


$(8-4\sqrt{2}, 135°)$
0
4
8
12
16
$(-8-4\sqrt{2}, 135°)$
$\theta = 135°$
$r = \frac{4}{1+\sin\theta}$

The related equations are $r = -\frac{4}{1 - \sin\theta}$ and $\theta = 315°$.

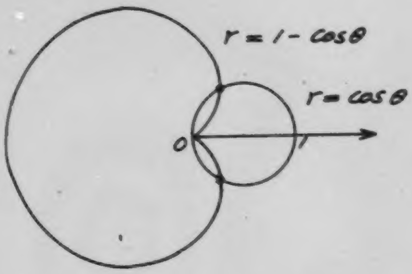
3. $(0,\theta)$ and $(\frac{\sqrt{2}}{2}, 45°)$

Related equations are the same.



$r = 2\sin\theta$
$r = 2\cos\theta$
0
1
2

4. $(0,\theta)$ $(\frac{1}{2}, 60°)$ $(\frac{1}{2}, 300°)$

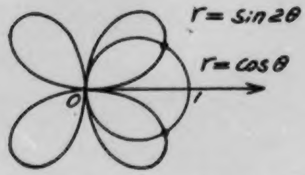


$r = 1 - \cos\theta$
$r = \cos\theta$
0
1

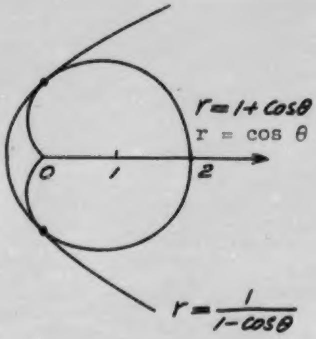
The related equations are
$r = \cos\theta$ and $r = -1 - \cos\theta$

5. $(0,\theta)$ $(\frac{\sqrt{3}}{2}, 30^\circ)$ $(\frac{\sqrt{3}}{2}, 150^\circ)$

7. $(1,90^\circ)$ $(1,270^\circ)$



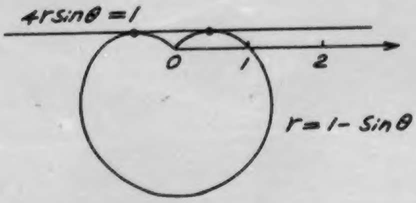
The related equations are $r = \cos\theta$ and
$r = -\sin 2\theta$



$r = \cos\theta - 1$

$r = \frac{-1}{1 + \cos\theta}$

6. $(\frac{1}{2}, 30^\circ)$ $(\frac{1}{2}, 150^\circ)$



The related equations are $-1 - \sin\theta$ and
$4r\sin\theta = 1$.

6-7. Families of Curves.

This topic is a necessary foundation for parts of the usual calculus courses. We expand the idea of linear combinations of functions and treat families of curves more generally. A one-parameter family is related to the physical concept of one "degree of freedom" and so on. It is instructive to develop this briefly, showing how the restriction of each degree of freedom is equivalent to the assignment of a specific value to one parameter. Thus, drawing a circle on the blackboard involves three degrees of freedom: locate it horizontally, locate it vertically, determine its radius. These determinations are made, and the degrees of freedom are restricted, by assigning specific values respectively to a, b, r, in the three-parameter family: $(x - a)^2 + (y - b)^2 = r^2$.

The method of choosing a particular member of a family is often more complicated than assigning numerical values to the parameters. The particular member may be determined by a condition whose application may be quite indirect. There will be future applications of the method shown for picking out a tangent line, by imposing the condition that such a line has a double contact point. Algebraically this means that the equation which gives the abscissas of the intersection points must have multiple roots. If the equation is quadratic, as in the text, this condition means that the discriminant of the quadratic equation must equal zero. This is what gives us the equation from which we pick out the values of the parameter for the members of the family that satisfy our condition.

The review exercises at the end of the chapter furnish many opportunities to write families of curves, mostly lines and circles. We did not include exercises in which the student is asked to pick out a particular member of the family to satisfy a given condition. These are simple to improvise, and may take any of the following forms: Find the member of the family which goes through a given point; find the member of the family that has a given slope; find the member of the family that is tangent to a given circle; etc. It is instructive to consider each family as a set of curves; then the question of finding a particular curve that satisfies two conditions is equivalent to the question of finding the intersection of two sets of curves. Note carefully that we use the word "intersection" here to mean the curve (or curves) common to the two sets of curves that comprise the two families.

Exercises 6-7

1. $x = a$.
2. $y = a$.
3. $y + 1 = k(x - 2)$.
4. $y = kx + b$.
5. $(x + 1)^2 + (y - 2)^2 = a^2$.
6. $(x - a)^2 + (y - b)^2 = 16$.
7. $\frac{x}{a} = y^2$.
8. $3x - 4y + k = 0$.
9. $x - 2y + k = 0$.
10. $x \cos t + y \sin t - 5 = 0$.
11. $x \cos t + y \sin t - p = 0$, $p > 5$.
12. $x^2 + y^2 - 2hx - 2ky = 0$, $h^2 + k^2 = 36$; or

    $x^2 + y^2 - 2hx - 2\sqrt{36 - h^2}\, y = 0$.

13. $(x - h)^2 + (y - k)^2 = 1$, $h^2 + k^2 > 1$.

14. $y = 4x - 12$.

15. $3x - 3y + 20 = 0$.

16. $4x + 5 = 0$.

17. The family of all lines through the intersection of the given lines is represented $a(x - 2y + 3) + b(x + 3y - 2) = 0$. Picking a and b so that $(1,1)$ lies on the line and simplifying we get $y = 1$.

18. $(a + b)x^2 + (a + b)y^2 - 2(a - b)x + 4by - 35a - 44b = 0$
For each pair (a,b) this equation represents a circle through the intersection of the given two. The whole family of circles is called a coaxial family. Their centers are all on the perpendicular bisector of the common chord of any two of these circles. If $a = 1$, $b = 0$ we get the first circle; if $a = 0$, $b = 1$, we get the second circle; if $a = -b$ we get a line, the line along this common chord. This line may be considered a degenerate circle. It has the property that from any point on it the tangents to all members of the coaxial family have equal lengths.

19. $x + 3y - 7 = 0$.

20. $3x + 4y - 15 = 0$.

21. $y - 5 = \frac{-72}{a^2}(x - 2)$ a represents the x-intercept: $\frac{72}{a}$, the y-intercept. Thus $\frac{-72}{a^2}$ is the slope, and $\frac{1}{2}(a)(\frac{72}{a})$ the area of the triangle in the first quadrant.

22. $x = 5$ and $3x - 4y + 25 = 0$. Any line through the intersection of the given lines can be represented by $a(y - 10) + b(2x - y) = 0$, that is, by $2bx + (a - b)y - 10a = 0$. The distance from such a line to the origin is $\frac{|2b(0) + (a - b)0 - 10a|}{\sqrt{4b^2 + (a - b)^2}}$ that is $\frac{|10a|}{\sqrt{a^2 - 2ab + 5b^2}}$.

If this distance is 5 then $|2a| = \sqrt{a^2 - 2ab + 5b^2}$. Thus $3a^2 + 2ab - 5b^2 = 0$. Set $a = tb$ and we get $3t^2 + 2t - 5 = 0$ and $t = 1$ or $-\frac{5}{3}$. These give us the solutions above.

Review Exercises

1. First, we find the coordinates of the intersections of the two given lines with all lines parallel to the x-axis. For a line parallel to the x-axis, the y-coordinates of the two intersections are the same. Thus we have $x_1 - 8 = -y_1$ and $2x_2 - 1 = y_2 = y_1$, or $x_2 - \frac{1}{2} = \frac{y_1}{2}$. Adding gives $\frac{x_1 + x_2}{2} = \frac{17 - y_1}{4}$. Hence the equation of the desired locus is $x = \frac{17 - y}{4}$ or $y = -4x + 17$.

2. In the same manner as in 1 above we find the equation

$$y = \frac{1}{2}x + \frac{7}{2}$$

3. (a) $d(P,A) = \sqrt{(x + 4)^2 + y^2} = 2d(P,B) = 2\sqrt{(x - 4)^2 + y^2}$ which gives us upon simplifictaion $3x^2 + 3y^2 - 40x + 48 = 0$ which is the equation of a circle.

   (b) $\sqrt{(x + 4)^2 + y^2} + \sqrt{(x - 4)^2 + y^2} = 10$ which gives $\frac{x^2}{25} + \frac{y^2}{9} = 1$ which is the equation of an ellipse.

   (c) $\sqrt{(x + 4)^2 + y^2} - \sqrt{(x - 4)^2 + y^2} = 2$ which gives $x^2 - \frac{y^2}{15} = 1$ which is the equation of a hyperbola.

   (d) If the lines are perpendicular, the product of their slopes is -1, therefore

   $\frac{y - 0}{x + 4} \cdot \frac{y - 0}{x - 4} = -1$, $\therefore y^2 = 16 - x^2$, or $x^2 + y^2 = 16$.

   This is an equation of a circle.

   (e) $\frac{y - 0}{x + 4} = 2 \cdot \frac{y - 0}{x - 4}$, $\therefore y = 0$; or $x = -12$.

   The locus is the pair of lines whose equations are given above.

   (f) $\frac{y - 0}{x + 4} = 1 + \frac{y - 0}{x - 4}$, therefore $xy - 4y = x^2 - 16 + xy + 4y$.

   This equation may be written $x^2 + 8y - 16 = 0$, and is an equation of a parabola.

(g) Let α be the inclination of $\overleftrightarrow{PA}$ and β be the inclination of $\overleftrightarrow{PB}$. If m_1 , m_2 are their respective slopes then

$\tan \angle APB = \tan(\beta - \alpha)$.

$$\therefore 1 = \frac{m_2 - m_1}{1 + m_2 m_1} \quad \text{or} \quad 1 = \frac{\frac{y}{x+4} - \frac{y}{x-4}}{1 + \frac{y}{x+4} \cdot \frac{y}{x-4}} = \frac{8y}{x^2 - 16 + y^2} .$$

This equation may be written more simply as $x^2 + (y - 4)^2 = 32$, and is an equation of a circle.

(h) In the same way as in (g) above, we have

$\tan \angle APB = \tan 60^\circ = \sqrt{3} = \frac{m_2 - m_1}{1 + m_2 m_1} = \frac{8y}{x^2 - 16 + y^2}$. This equation

may be written $x^2 + y^2 - \frac{8}{\sqrt{3}} y = 16$ and is an equation of a circle.

(i) Area = $\frac{1}{2}hb$; $d(A,B) = 8$ $\therefore$ $\frac{1}{2}8h = 20$ $\therefore$ $h = 5$.

Since the distance from P to the x-axis must be 5 , the locus of P is the pair of lines whose equations are $y = +5$, $y = -5$.

(j) $\sqrt{(x + 4)^2 + y^2} < \sqrt{(x - 4)^2 + y^2}$ means that $x < 0$.

4. Let $M = (r,s)$ be the midpoint of $\overline{AP}$. Then $r = \frac{x + 6}{2}$, $s = \frac{y + 0}{2}$;

$\therefore$ $x = 2r - 6$ and $y = 2s$. Since $P = (x,y)$ is on the circle $x^2 + y^2 = 36$, we have $(2r - 6)^2 + (2s)^2 = 36$ or $r^2 - 6r + 9 + s^2 = 9$. This may be written $r^2 + s^2 - 6r = 0$ or $x^2 + y^2 - 6x = 0$ and is an equation of a circle, which is the required locus.

5. Let $P = (r,s)$. Then $x = \frac{r + 0}{2}$, $y = \frac{s + 5}{2}$. Since $x^2 + y^2 = 25$,

$\therefore$ $(\frac{r}{2})^2 + (\frac{s + 5}{2})^2 = 25$, or $r^2 + (s + 5)^2 = 100$. This equation may be written $x^2 + y^2 + 10y - 75 = 0$ and is an equation of a circle, which is the required locus.

6. Let $D = (x,y)$, $E = (20,t)$ and M , the midpoint of $\overline{DE} = (r,s)$. From similar triangles, $\frac{y}{x + 10} = \frac{t}{30}$, $\therefore$ $t = \frac{30y}{x + 10}$. We have $r = \frac{x + 20}{2}$, $s = \frac{y + t}{2}$; $\therefore$ $x = 2r - 20$, $y = 2s - t = 2s - \frac{30y}{x + 10}$. These equations yield $x = 2(r - 10)$, $y = \frac{2s(r - 5)}{r + 10}$. Since D is on the circle $x^2 + y^2 = 100$, we have $(2(r - 10))^2 + (\frac{2s(r - 5)}{r + 10})^2 = 100$, or, $(r - 10)^2 + \frac{s^2(r - 5)^2}{(r + 10)^2} = 25$. This may be written

$(r - 10)^2(r^2 - 20r + 100 - 25) + s^2(r - 5)^2 = 0$, or

$(r + 10)^2(r - 5)(r - 15) + s^2(r - 5)^2 = 0$; which is equivalent to

$(r - 15)(r + 10)^2 + (r - 5)s^2 = 0$. Therefore an equation for the required locus is $(x - 15)(x + 10)^2 + (x - 5)y^2 = 0$.

However, a much simpler solution is available in polar coordinates. Take the pole at C and the polar axis to the right along the x-axis. Let $D = (p,\theta)$, $E = (q,\theta)$, and M , the midpoint of $\overline{DE} = (r,\theta)$. Then $\frac{p}{20} = \cos\theta$, $\frac{30}{q} = \cos\theta$ and $r = \frac{1}{2}(p + q) = \frac{1}{2}(20 \cos\theta + 30 \sec\theta)$. Therefore an equation for the required locus is $r = 10 \cos\theta + 15 \sec\theta$. We may show the equivalence of these two solutions by using the relationship: $r^2 = x'^2 + y'^2$, $\cos\theta = \frac{x'}{\sqrt{x'^2 + y'^2}}$; and $x' = x + 5$, $y^1 = 5$. The computation is elementary but tedious.

7. Any line parallel to $y = 3x + 5$ has an equation $y = 3x + d$, and will intersect the circle $x^2 + y^2 - 4x + 8y = 0$ in two points whose abscissas are the roots of $x^2 + (3x + d)^2 - 4x + 8(3x + d) = 0$. If the midpoint of this chord has coordinates (r,s) , then $r = \frac{1}{2}$ the sum of the abscissas of the endpoint, that is, $\frac{1}{2}$ the sum of the roots of this equation, and this result can be found from the coefficients directly without solving the equation. Thus $10x^2 + (20 + 6d)x + d^2 + 8d = 0$, and $r = -\frac{10 + 3d}{10}$, $\therefore$ $s = 3r + d = -\frac{30 - d}{10}$. Eliminating d from these two equations yields $r + 3s + 10 = 0$, therefore an equation for the required locus is $x + 3y + 10 = 0$.

8. In the same manner as in Exercise 7, we find $x - 9y = 0$ as an equation for the locus.

9. (a) The line $\frac{x}{a} + \frac{y}{b} = 1$ has intercepts a and b. The conditions of the problem requires that $ab = \pm 24$. Therefore $\frac{x}{a} \pm \frac{ay}{24} = 1$ is a pair of equations representing two one-parameter families of lines, the solution we require. Of course $a \neq 0$.

   (b) As in 9(a), we need $\frac{x}{a} + \frac{y}{b} = 1$ and $a + b = 6$, that is $\frac{x}{a} + \frac{y}{6 - a} = 1$ with $a \neq 0, 6$. We may consider that a line parallel to an axis has just one intercept whose "sum" is itself; in which case we may include in our solution the lines $x = 6$; and $y = 6$.

   (c) $(x - a)^2 + (y - b)^2 = a^2$

   (d) $(x - a)^2 + (y - b)^2 = b^2$

   (e) The distance from the center (a,b) of one such circle to the line $4x + 3y - 2 = 0$ is $\frac{4a + 3b - 2}{5}$, and, by the conditions of the problem, equals ± 1. The centers must lie therefore on the lines $4x + 3y - 7 = 0$ and $4x + 3y + 3 = 0$, which are parallel to the original lines. The families of circles are therefore $(x - a)^2 + (y - b)^2 = 1$ where (a,b) must satisfy one of the equations of the lines just found. In terms of a single parameter the answers are $(x - a)^2 + (y - \frac{7}{3} + \frac{4}{3}a)^2 = 1$, and $(x - a)^2 + (y + 1 + \frac{4}{3}a)^2 = 1$.

   (f) The two families are $(x - a)^2 + (y - \frac{5r + 2}{3} + \frac{4a}{3})^2 = r^2$, $(x - b)^2 + (y + \frac{5r - 2}{3} + \frac{4}{3}b)^2 = r^2$.

   (g) $(x - a)^2 + (y - b)^2 = 36$ where $a^2 + b^2 < 36$.

   (h) $(x - a)^2 + (y - b)^2 = a^2 + b^2$.

   (i) The distance from $P = (x,y)$ on the circle to the center (a,b) must equal the distance from $(12,5)$ to the same center. Therefore the circles we want have equation:

$$(x - a)^2 + (y - b)^2 = (12 - a)^2 + (5 - b)^2 .$$

47

(j) $(x - a)^2 + (y - b)^2 = r^2$ where $a^2 + b^2 < r^2$.

(k) $(x - a)^2 + (y - b)^2 = 25$ where $a^2 + b^2 > 25$.

(ℓ) The two families are $(x - g)^2 + \left(y - \frac{d\sqrt{a^2 + b^2} - c - ag}{b}\right)^2 = d^2$

and $(x - h)^2 + \left(y + \frac{d\sqrt{a^2 + b^2} + c + ah}{b}\right)^2 = d^2$ where g and h are arbitrary.

(m) A point (a,b) on a bisector of the angles formed by the two lines must be equidistant from them, therefore

$\left|\frac{3a - 4b + 5}{5}\right| = \left|\frac{4a - 3b + 9}{5}\right|$. These bisectors have, therefore, the equations $3x - 4y + 5 = 4x - 3y + 9$; and $3x - 4y + 5 = -4x + 3y - 9$; that is, $x + y + 4 = 0$, and $x - y + 2 = 0$.

(Note that these lines are perpendicular to each other.)

Therefore $b = -a - 4$, or $b = a + 2$. The families of circles:

$(x - a)^2 + (y - b)^2 = r^2$, become

$(x - a)^2 + (y + a + 4)^2 = (\frac{7}{5}a + \frac{21}{5})^2$; and

$(x - a)^2 + (y - a - 2)^2 = (\frac{1}{5}a + \frac{3}{5})^2$.

(n) The families are $(x - g)^2 + (y - h)^2 = \frac{(a_1g + b_1h + c_1)^2}{a_1^2 + b_1^2}$ where

$$h = \frac{\dfrac{a_2g + c_2}{\sqrt{a_2^2 + b_2^2}} - \dfrac{a_1g + c_1}{\sqrt{a_1^2 + b_1^2}}}{\dfrac{b_1}{\sqrt{a_1^2 + b_1^2}} - \dfrac{b_2}{\sqrt{a_2^2 + b_2^2}}}$$

and for the other family

$$h = -\frac{\dfrac{a_2g + c_2}{\sqrt{a_2^2 + b_2^2}} + \dfrac{a_1g + c_1}{\sqrt{a_1^2 + b_1^2}}}{\dfrac{b_1}{\sqrt{a_1^2 + b_1^2}} + \dfrac{b_2}{\sqrt{a_2^2 + b_2^2}}}.$$

(o) $(x - a)^2 + (y - b)^2 = r^2$ where $b^2 \leq r^2$.

(p) $(x - a)^2 + (y - b)^2 = r^2$ where $a^2 > r^2$.

(q) $(x - g)^2 + (y - h)^2 = r^2$ where $\dfrac{|ga + hb + c|}{\sqrt{a^2 + b^2}} > r$.

(r) $(x - a)^2 + (y - b)^2 = r^2$ where $0 < r$, and $\sqrt{a^2 + b^2} + r < 10$.

(s) $(x - a)^2 + (y - b)^2 = r^2$ where $r > 0$ and $|r - 1| \leq \sqrt{a^2 + b^2} \leq r + 1$.

(t) $ax + by + d = 0$ where $\dfrac{d}{\sqrt{a^2 + b^2}} \leq 1$.

(u) $(x - a)^2 + (y - b)^2 = r^2$ where $0 < a$, $0 < b$, $a + b < 10$, and $r <$ the smallest of $\{a, b, \dfrac{|a + b - 10|}{\sqrt{2}}\}$.

(v) $(x - a)^2 + (y - b)^2 = r^2$ where $r^2 > a^2 + (10 - b)^2$, $r^2 > a^2 + b^2$, and $r^2 > (a - 10)^2 + b^2$.

(w) $(x - a)^2 + (y - b)^2 = r^2$ where $\sqrt{a^2 + b^2} + r = 10$.

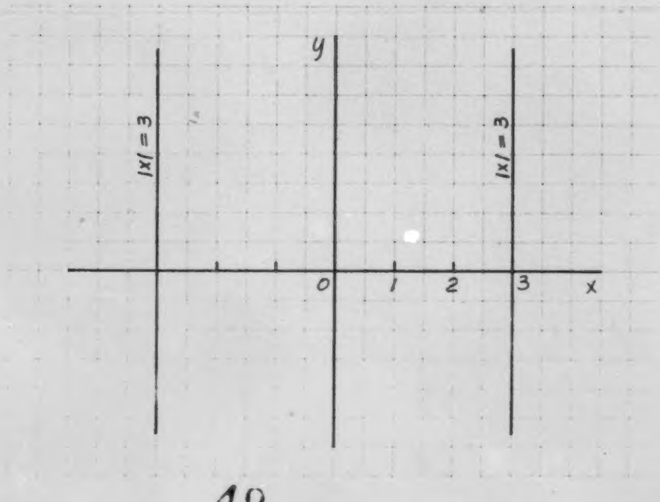
(x) $(x - a)^2 + (y - b)^2 = r^2$ where $\sqrt{a^2 + b^2} - r = 10$.

(y) $(x - a)^2 + (y - b)^2 = r^2$ where $r - \sqrt{a^2 + b^2} = 10$.

(z) $(x - g)^2 + (y - h)^2 = R^2$ where $\dfrac{|ag + bh + c|}{\sqrt{a^2 + b^2}} = R$ and

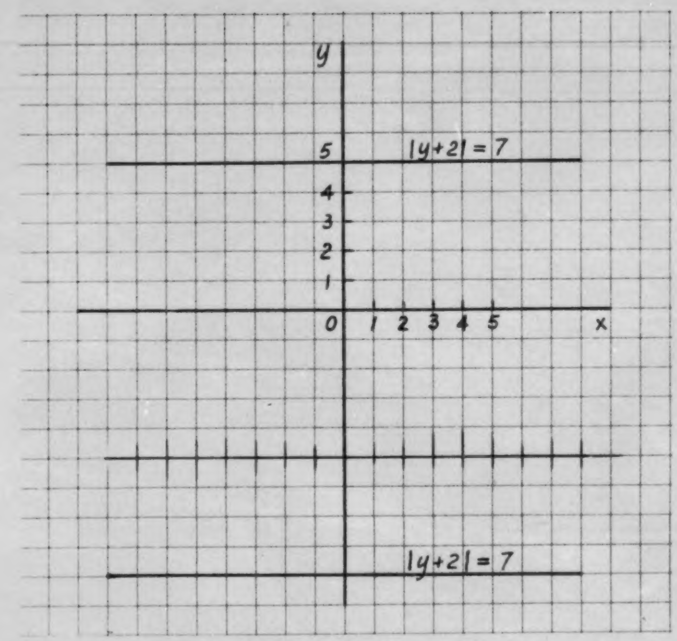
$(r - g)^2 + (s - h)^2 = R^2$.

10. (a)

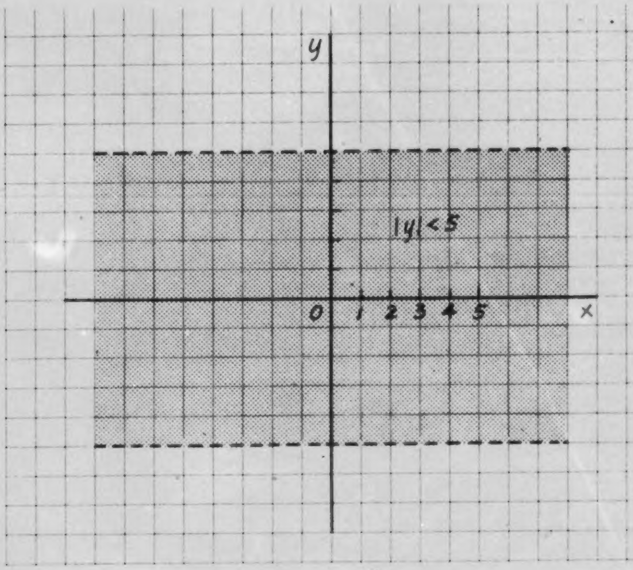


49

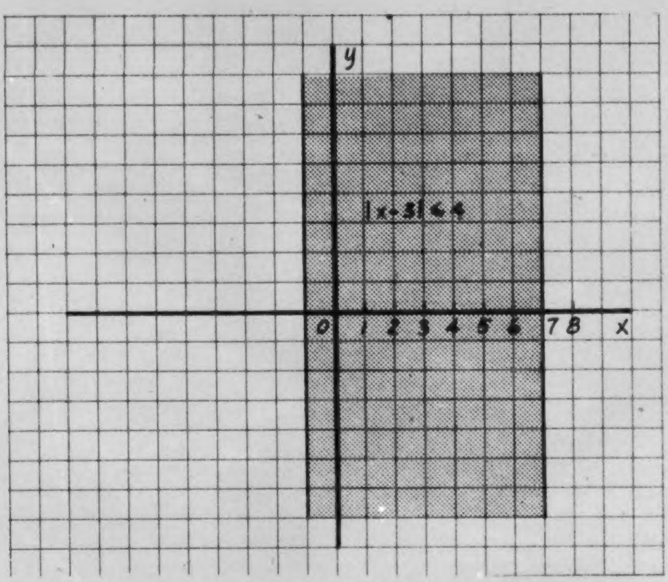
(b)



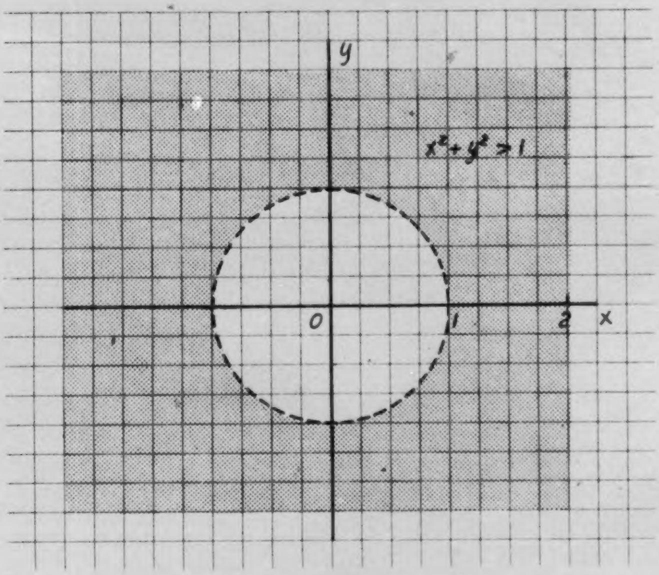
(c)



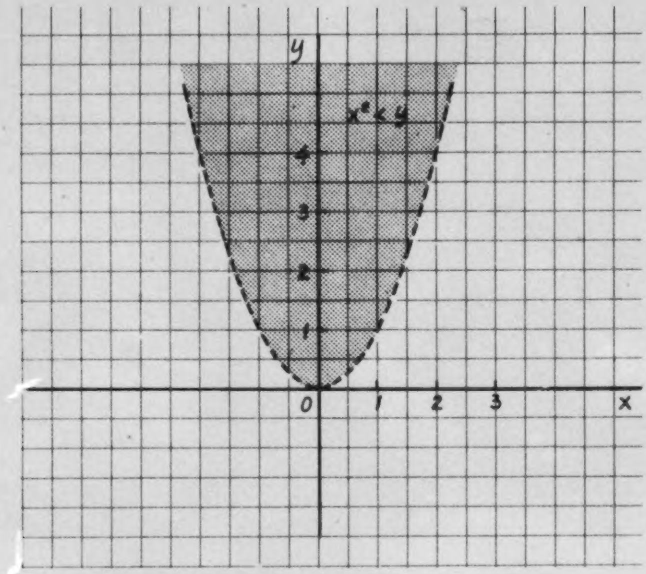
(a)



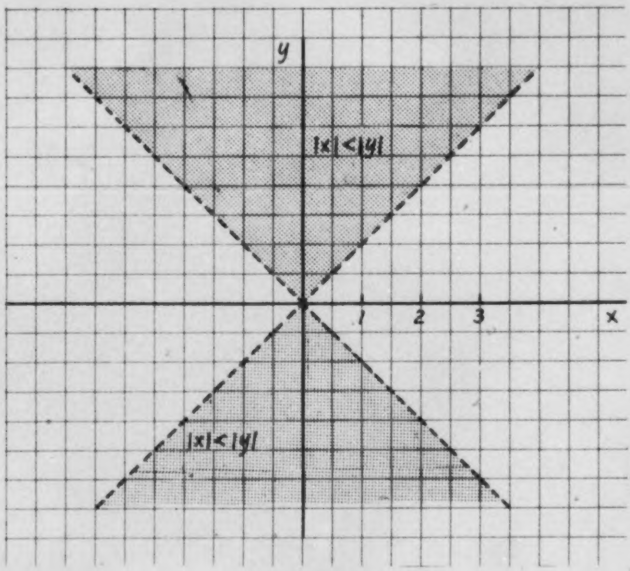
(e)



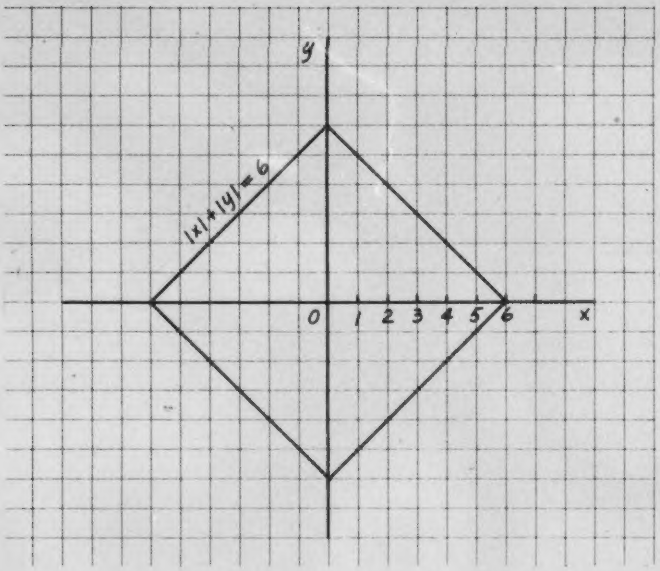
(f)



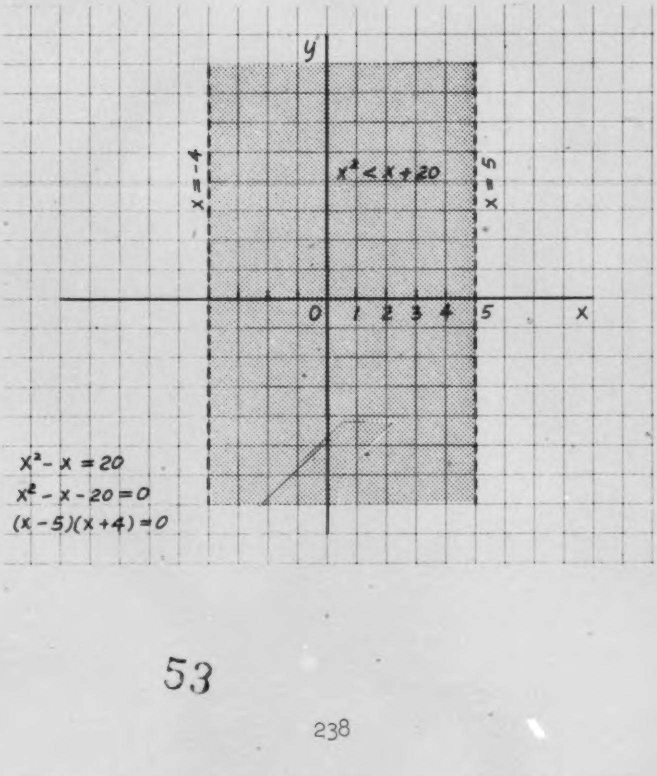
(g)



(h)



(i)



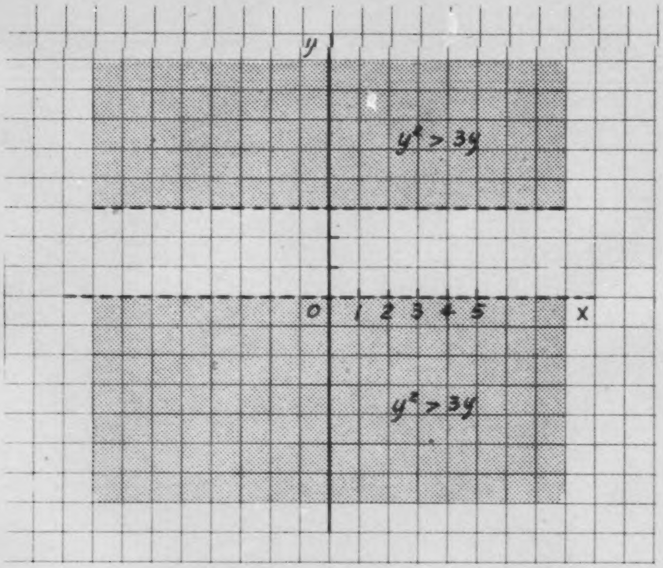
$x^2 - x = 20$

$x^2 - x - 20 = 0$

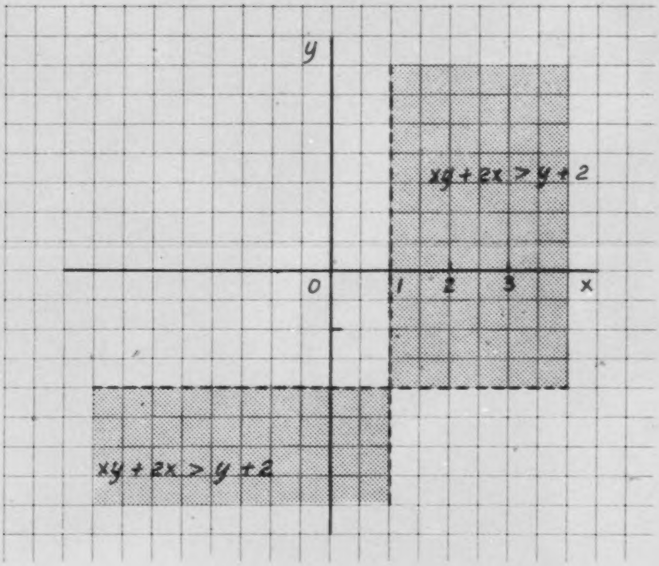
$(x-5)(x+4) = 0$

53

(j)

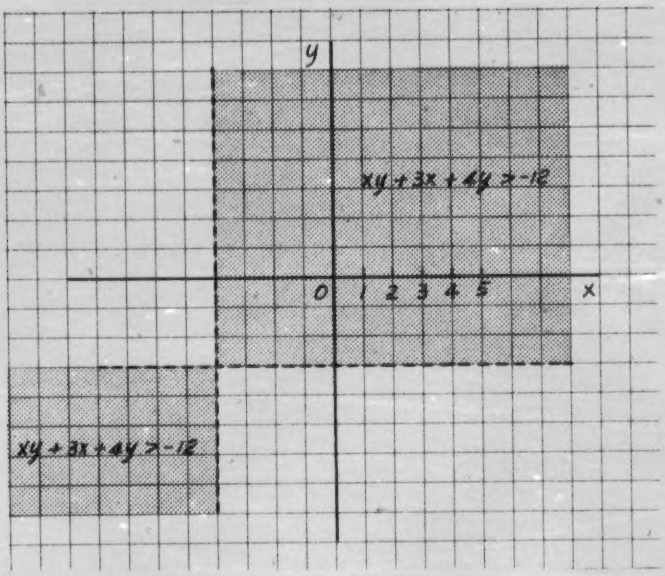


(k)

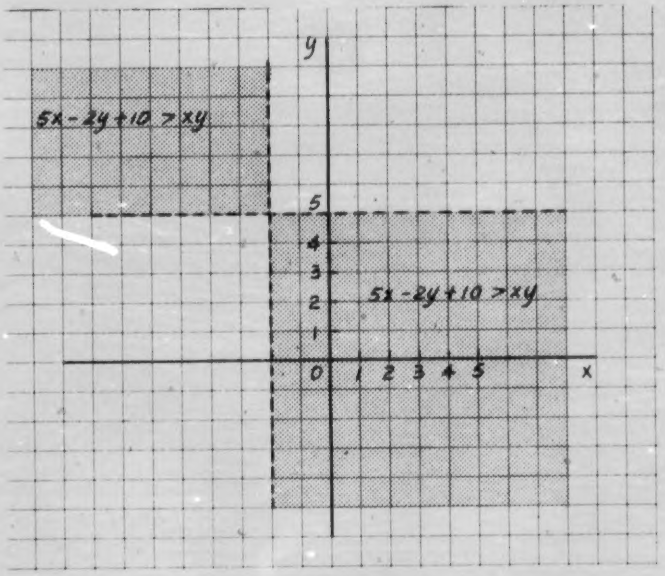


The given condition is equivalent to $xy + 2x - y - 2 > 0$ or $(x - 1)(y + 2) > 0$. Therefore both factors must be positive, or both factors must be negative. These conditions require $x > 1$ and $y > -2$; or $x < 1$ and $y < -2$.

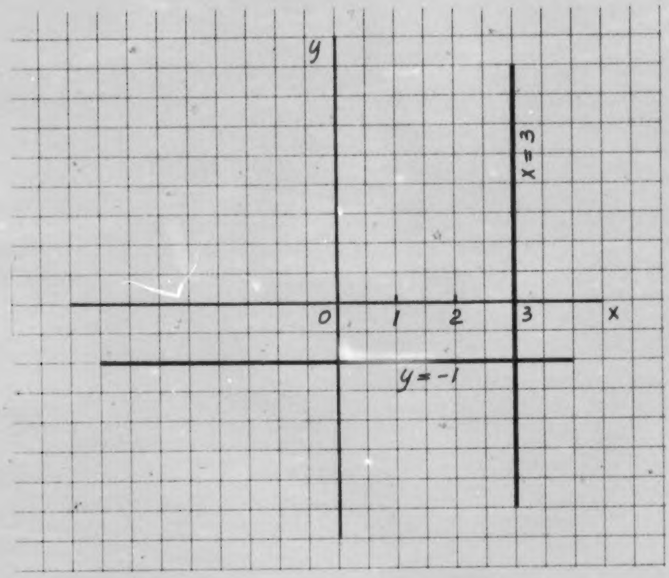
(ℓ)


y
xy + 3x + 4y > -12
0
1
2
3
4
5
x
xy + 3x + 4y > -12

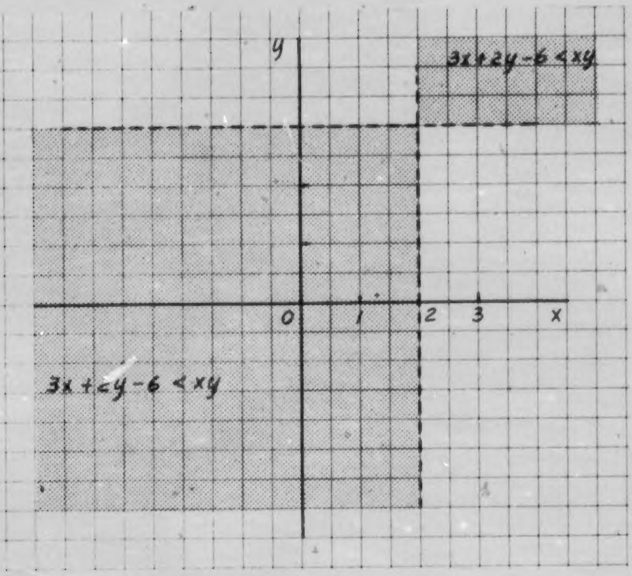
(m)


y
5x - 2y + 10 > xy
5
4
3
2
1
5x - 2y + 10 > xy
0
1
2
3
4
5
x

(n)

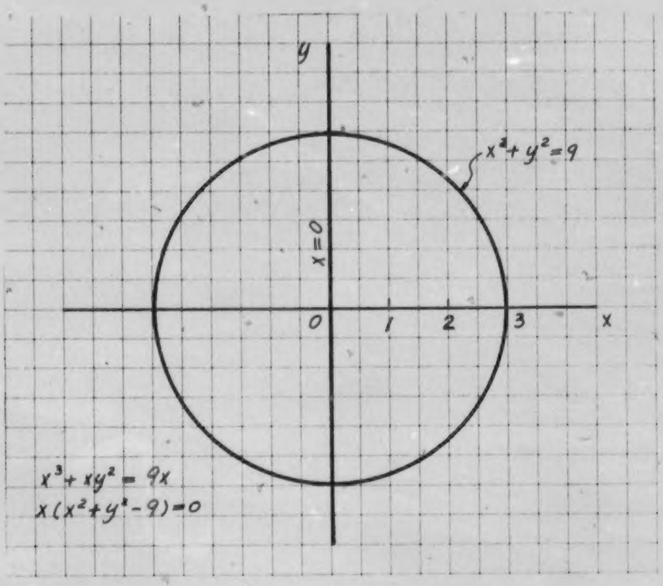


(o)



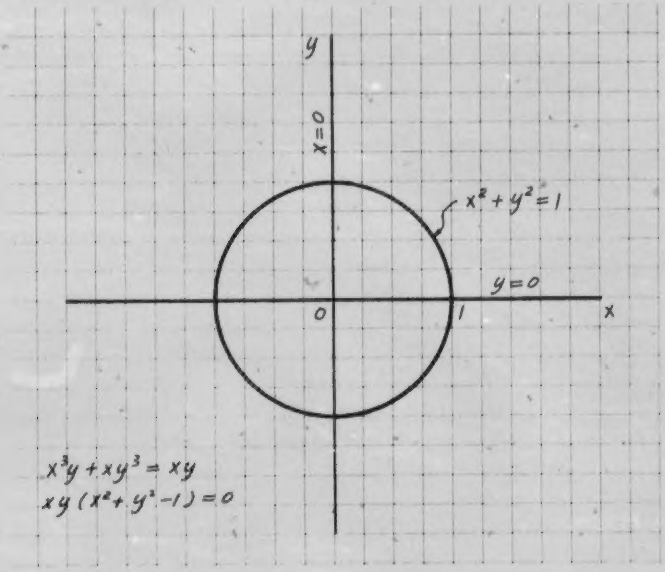
56

(p)



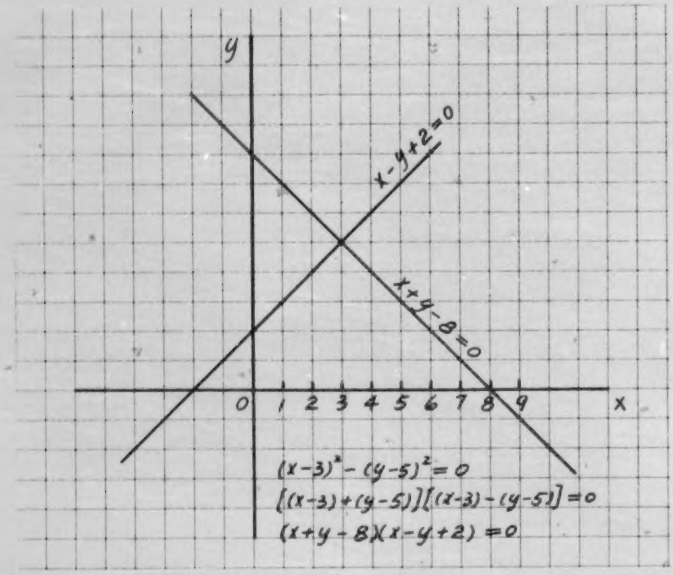
The graph is the set of all points on the circle, and all points on the y-axis, as shown.

(q)

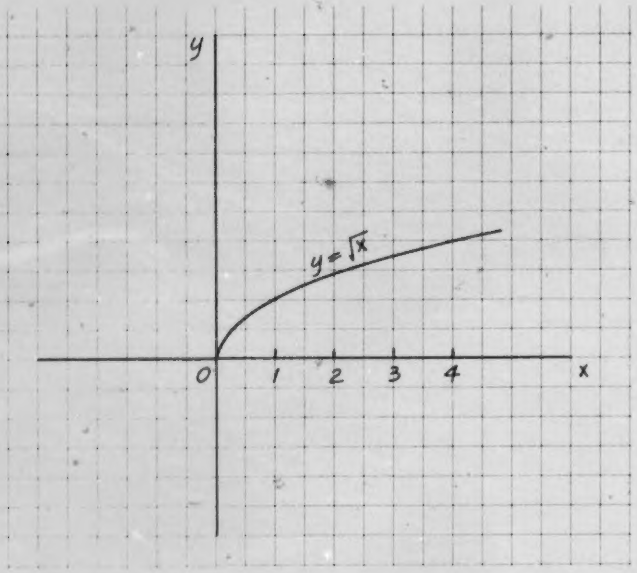


The graph is the set of all points on the circle, all points on the x-axis, and all points on the y-axis, as shown.

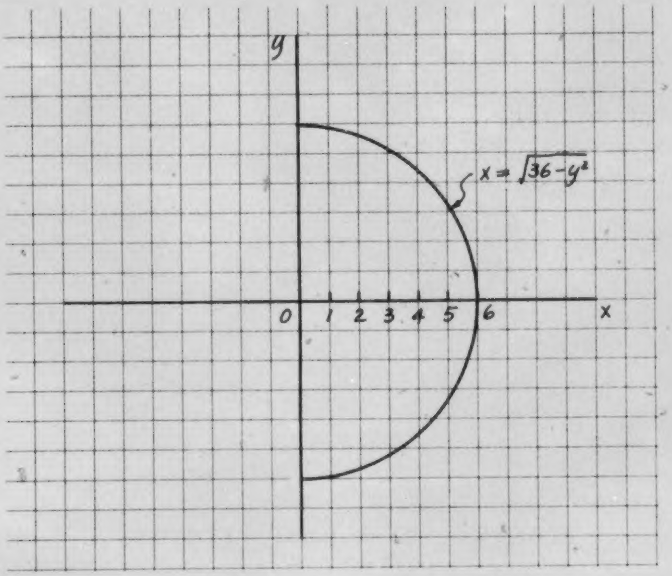
(r)


y
x-y+2=0
x+y-8=0
0
1
2
3
4
5
6
7
8
9
x
$(x-3)^2-(y-5)^2=0$
$[(x-3)+(y-5)][(x-3)-(y-5)]=0$
$(x+y-8)(x-y+2)=0$

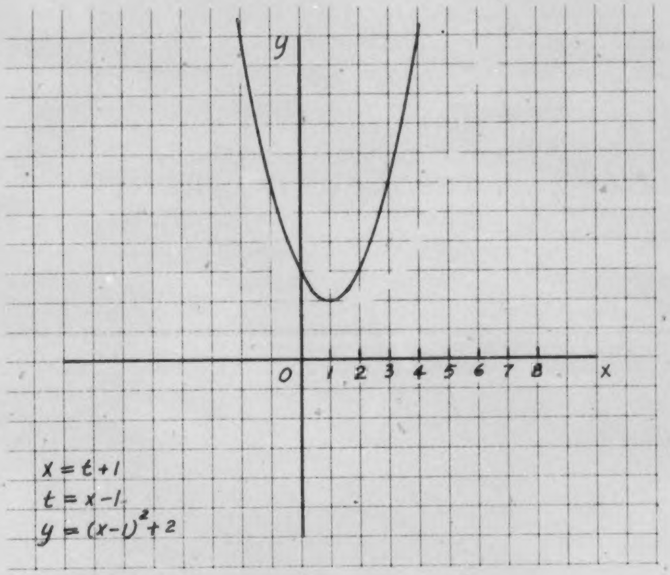
(s)


y
$y=\sqrt{x}$
0
1
2
3
4
x

(t)



11. (a)

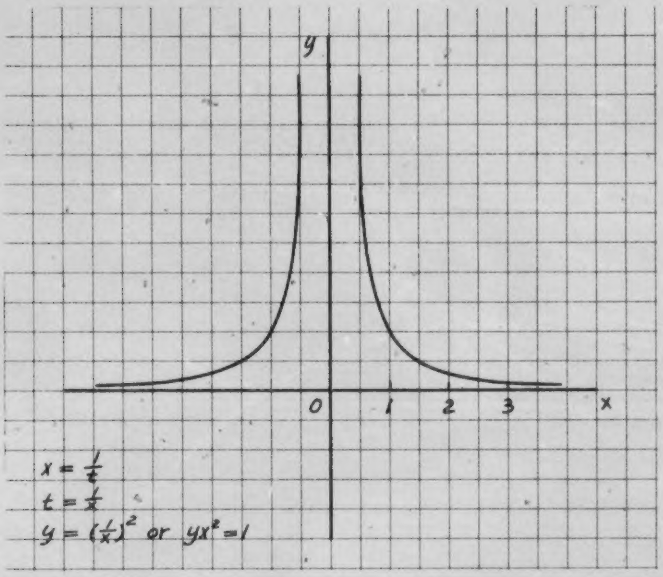


$x = t + 1$

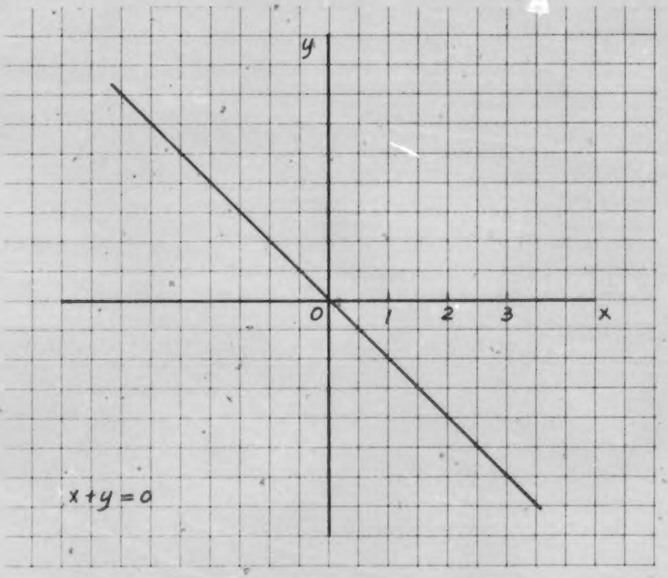
$t = x - 1$

$y = (x-1)^2 + 2$

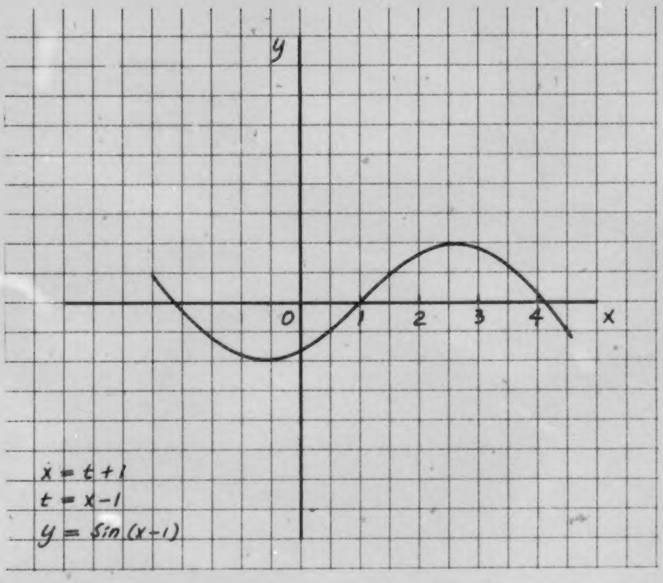
(b)



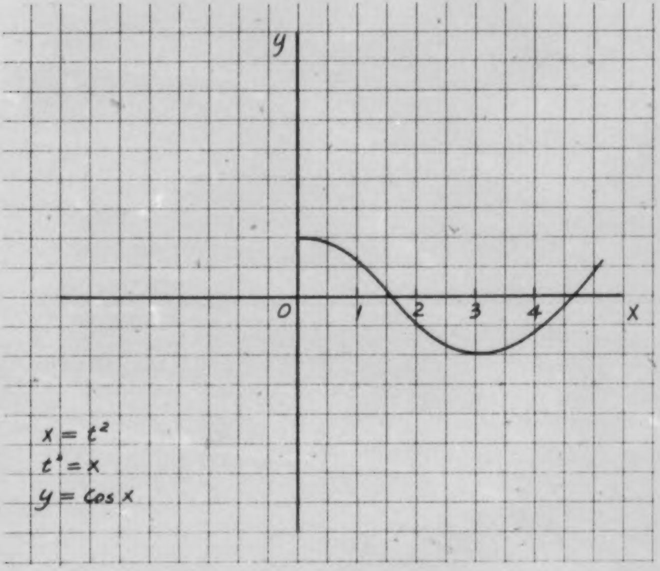
(c)



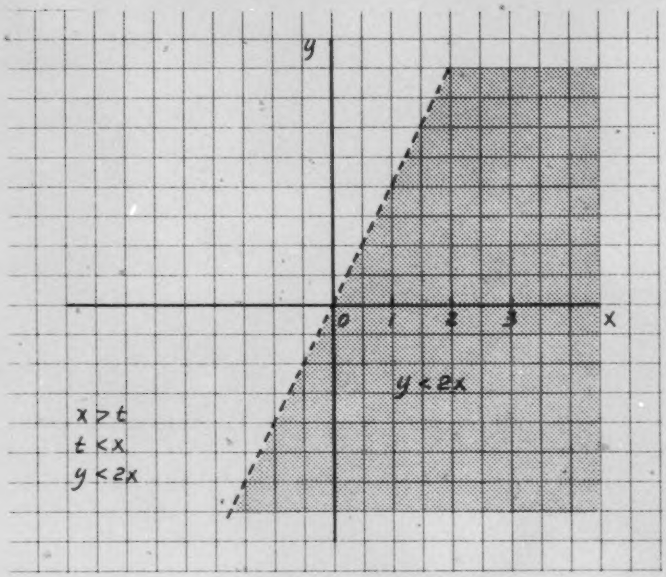
(d)



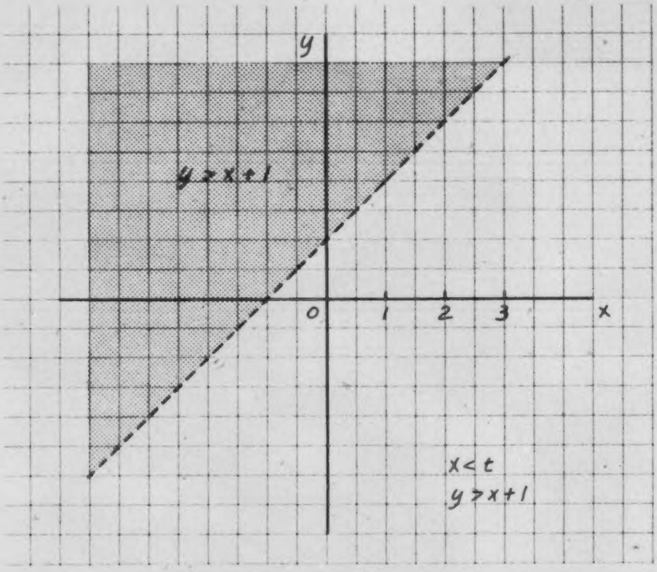
(e)



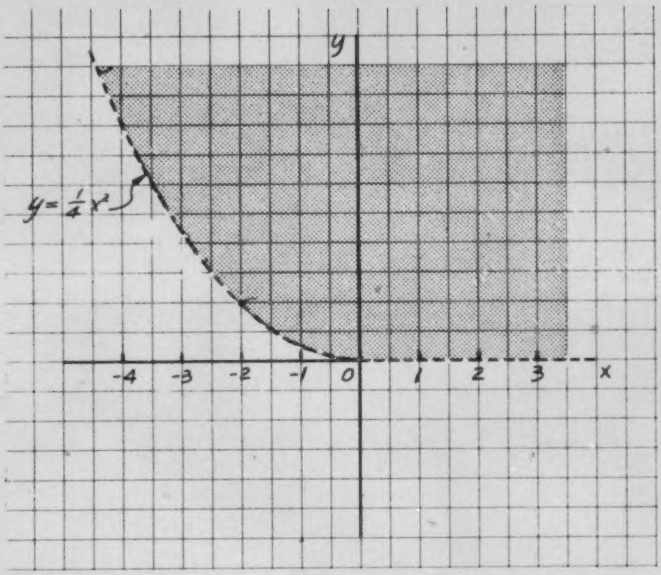
(f)



(g)

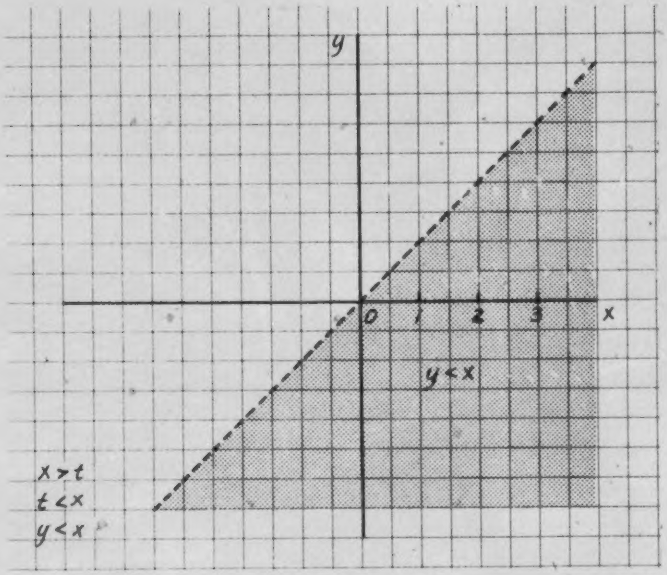


(h)

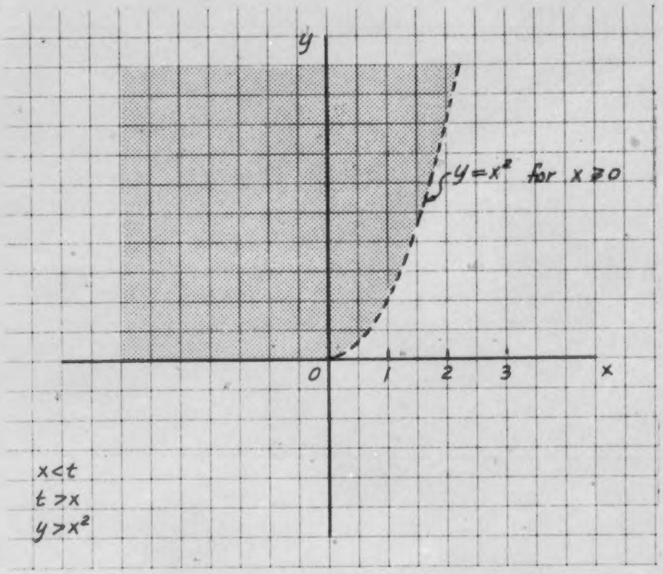


From $y = t^2$ we have $t = \pm\ y$ and $x > \pm 2\ y$. There $y \geq 0$, and our locus is the set of all points above the x-axis and to the right of either branch of the parabola $x^2 = 4\ y$. It is sufficient to take all points which are both above the x-axis, and to the right of the left branch, as shown.

(i)

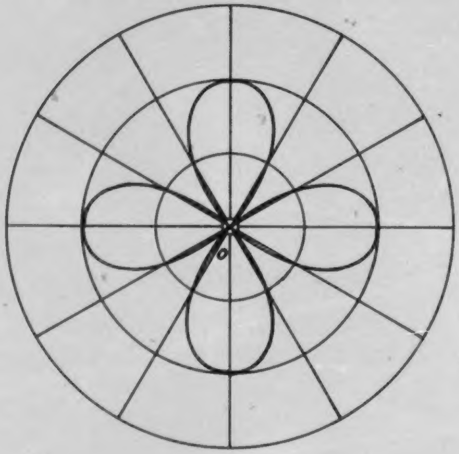


(j)



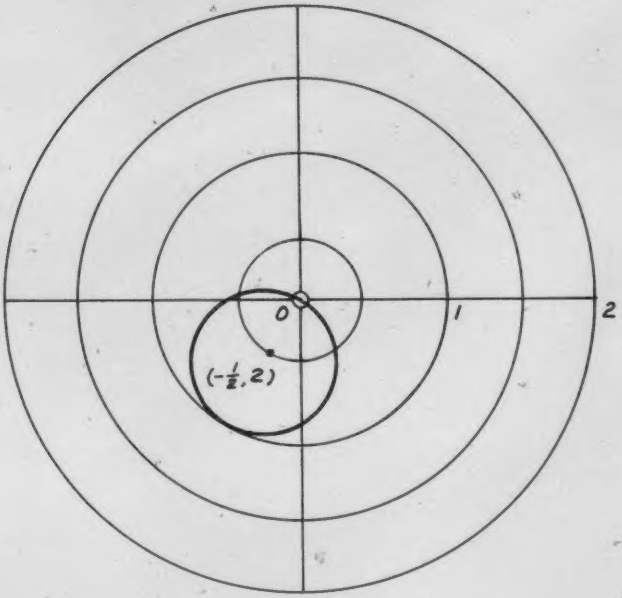
64

12. (a)



$r = \cos 2\theta$. Four leafed rose, symmetric with respect to the pole, and with respect to the lines $\theta = 0^\circ$, $\theta = 90^\circ$, $\theta = 45^\circ$, $\theta = 135^\circ$. Symmetry with respect to the last two lines can be shown if we use the related polar equation, thus: the points $(r, 45^\circ - \alpha)$, $(r, 45^\circ + \alpha)$ are symmetrically situated with respect to the line $\theta = 45^\circ$, but $\cos 2(45^\circ - \alpha) \neq \cos 2(45^\circ + \alpha)$. However, the point $(r, 45^\circ + \alpha)$ is on the curve for which we have the equation $r = -\cos 2(\theta - 180^\circ)$, and we now have $\cos 2(45^\circ - \alpha) = -\cos 2(45^\circ + \alpha + 180^\circ)$, as can easily be shown. In the same way we could show symmetry with respect to the line $\theta = 135^\circ$ by showing $\cos 2(135^\circ - \alpha) = -\cos 2(135^\circ + \alpha + 180^\circ)$.

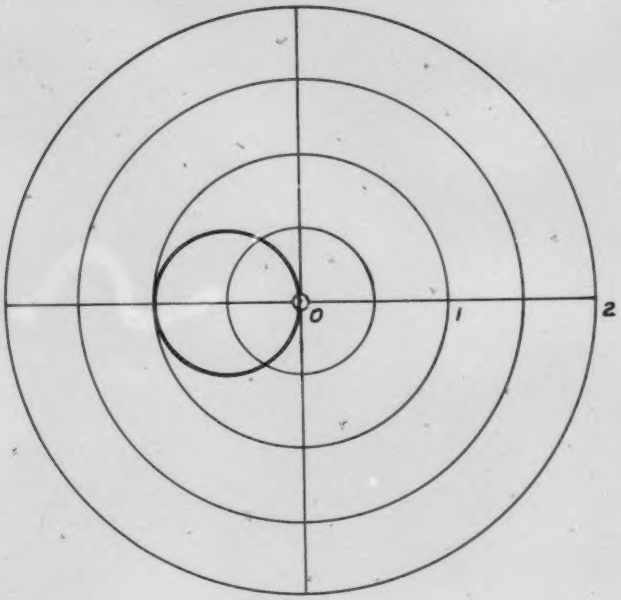
(b)


0
1
2
$(-\frac{1}{2}, 2)$

$r = \cos(\theta + 2)$

Circle of radius $\frac{1}{2}$, with center at $(\frac{1}{2}, -2)$.

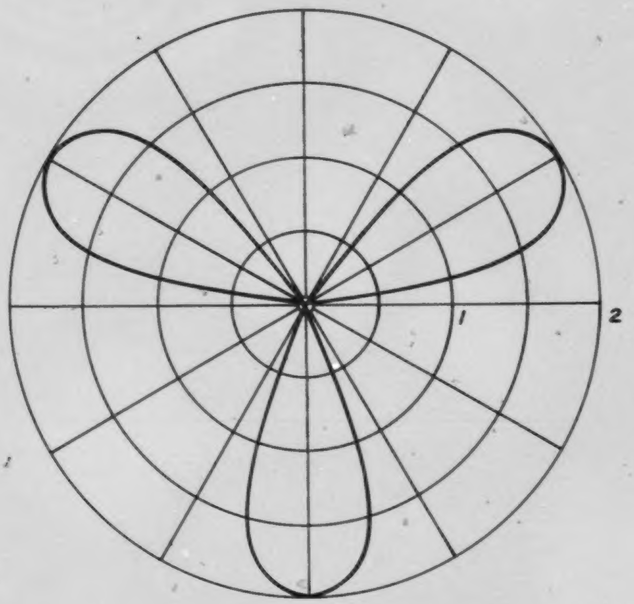
(c)



$r = \sin(\theta - \frac{\pi}{2}) = -\cos\theta$

Circle of radius $\frac{1}{2}$ with center at $(\frac{1}{2}, \pi)$.

(d)

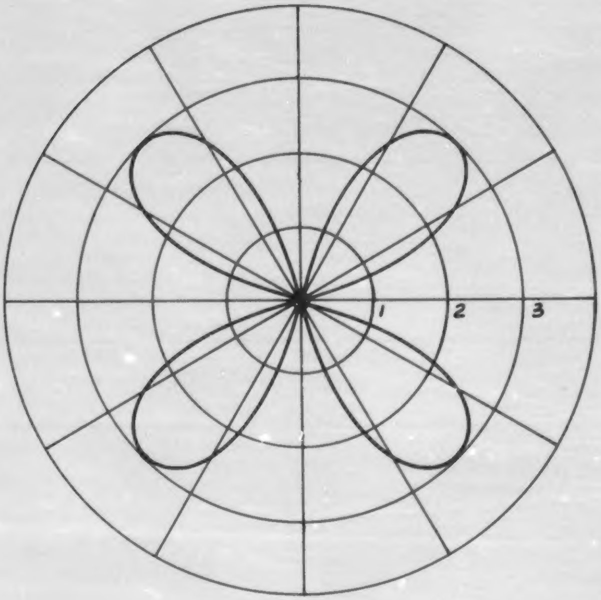


$r = 2 \sin 3\theta$

Three leafed rose symmetric with respect to $\theta = 30^\circ$, $\theta = 150^\circ$ and $\theta = 270^\circ$.

This last line has also the equation $\theta = 90^\circ$.

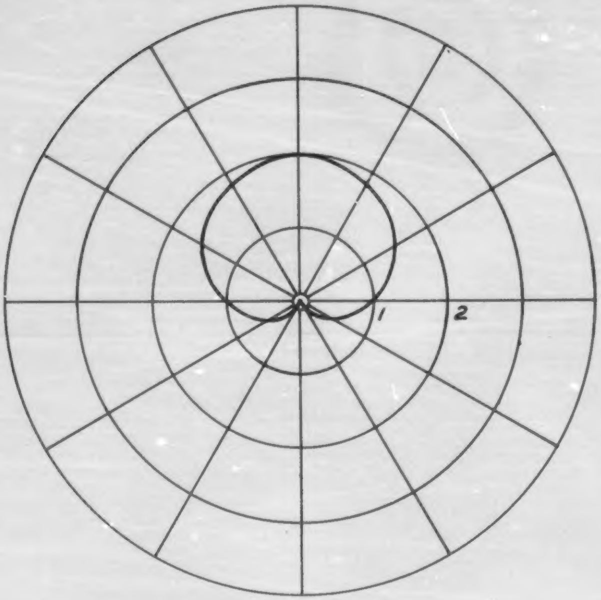
(e)



$r = 3 \sin 2\theta$

Four leafed rose symmetric with respect to $\theta = 45^\circ$, $\theta = 135^\circ$, $\theta = 0^\circ$ and $\theta = 90^\circ$. It is also symmetric with respect to the pole.

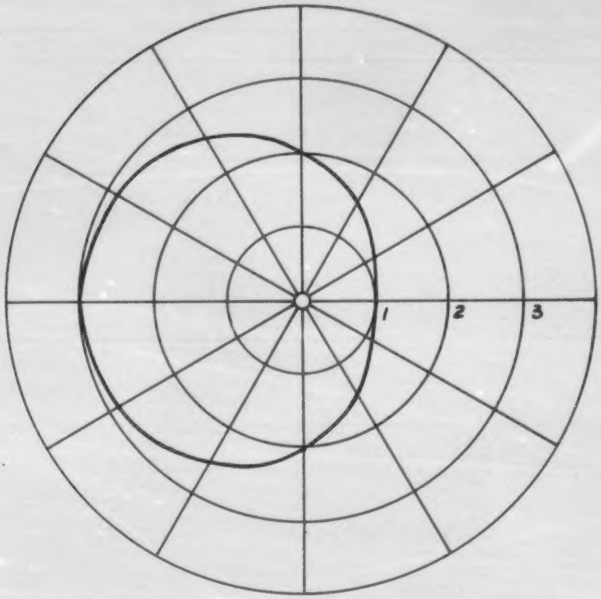
(f)



$r = 1 + \sin\theta$

Cardiod symmetric with respect to $\theta = 90^\circ$.

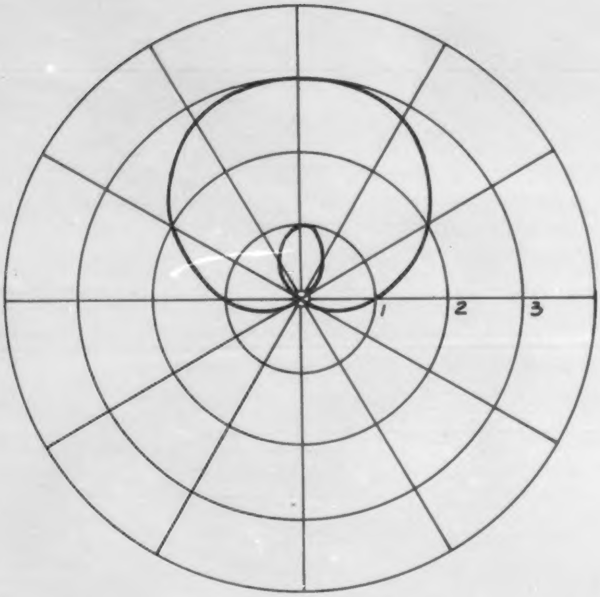
(g)



$r = 2 - \cos\theta$

Limacon symmetric with respect to $\theta = 0^\circ$.

(h)

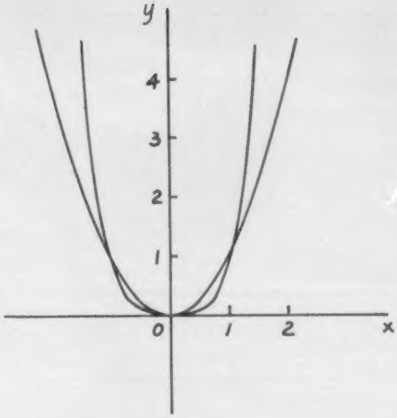


$r = 1 + 2 \sin\theta$

Limaçon symmetric with respect to $\theta = 90^\circ$

Review Exercises

13. $y = x^2 \quad y = x^4$

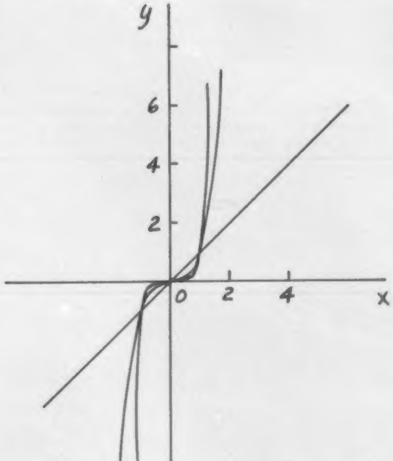


If $m > n$

for $|x| < 1$, $x^{2m} < x^{2n}$

for $|x| > 1$, $x^{2m} > x^{2n}$

14. $y = x \quad y = x^3 \quad y = x^5$

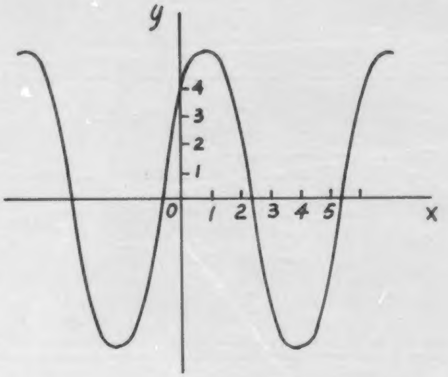


Generalization.

Let m , n be odd integers,
$0 < n < m$.
Then for

$x < -1$ we get	$x^m < x^n < -1$
$x = -1$	$x^m = x^n = -1$
$-1 < x < 0$	$-1 < x^n < x^m < 0$
$x = 0$	$x^n = x^m = 0$
$0 < x < 1$	$0 < x^m < x^n < 1$
$x = 1$	$x^m = x^n = 0$
$1 < x$	$1 < x^n < x^m$

15. $y = 3 \sin x + 4 \cos x$ $\qquad$ $y = 5(\frac{3}{5} \sin x + \frac{4}{5} \cos x)$



We know that $\sin(\alpha + \beta) = \sin\alpha \cos\beta + \sin\beta \cos\alpha$.

Let $\arccos \frac{3}{5} = \theta \approx 53^{\circ}$.

Then $\sin(x + \theta) = \frac{3}{5} \sin x + \frac{4}{5} \cos x$.

So $y = 5 \sin(x + \theta) \approx 5 \sin(x + 53^{\circ})$.

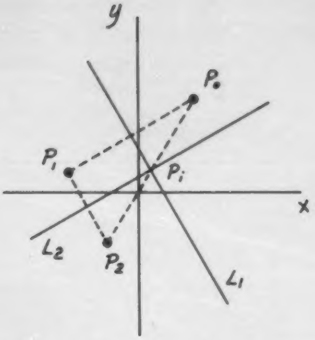
16. If $y = a \sin x + b \cos x$ we may write

$$y = \sqrt{a^2 + b^2}\left(\frac{a}{\sqrt{a^2 + b^2}} \sin x + \frac{b}{\sqrt{a^2 + b^2}} \cos x\right)$$

or letting $= \arccos \frac{a}{\sqrt{a^2 + b^2}}$

$$y = \sqrt{a^2 + b^2} \sin(x + \theta)$$

17. Let $L_1 : ax + by + c = 0$ and $L_2 : -bx + ay + d = 0$ be two perpendicular lines. Let S be a set symmetric with L_1 and L_2.



Denote the intersection of L_1 and L_2 by P_i.

$$P_i = \left(\frac{bd - ac}{a^2 + b^2}, \frac{-ad - bc}{a^2 + b^2}\right)$$

We need to show that for any P_0 in S, its reflection in P_i is also in S.

If $P_0 = (p_0, q_0)$ is in S, then the reflection $P_1 = (p_1, q_1)$ of P_0 in L_1 is still in S. Since $L_1 \perp L_2$, the point P_1 is determined by equations

$$(1) \qquad \frac{ap_1 + bq_1 + c}{\sqrt{a^2 + b^2}} = -\frac{ap_0 + bq_0 + c}{\sqrt{a^2 + b^2}}$$

$$(2) \qquad \frac{-bp_1 + aq_1 + d}{\sqrt{a^2 + b^2}} = \frac{-bp_0 + ap_0 + d}{\sqrt{a^2 + b^2}}$$

i.e., the conditions, $d(P_0, L_1) = d(P_1, L_1)$ and P_1, P_0 on opposite sides,
and $d(P_0, L_2) = d(P_1, L_2)$ and P_1, P_0 on same side.

Solving these we find

$$P_1 = (p_1, q_1) = \left(\frac{p_0(b^2 - a^2) - 2ac}{a^2 + b^2}, \frac{q_0(a^2 + b^2) - 2bc}{a^2 + b^2}\right)$$

Since P_1 is in S, the reflection P_2 of P_1 in L_2 is in S. Setting up and solving equations analogous to (1) and (2) we find

$$P_2 = (p_2, q_2) = \left(2\frac{(bd - ac)}{a^2 + b^2} - p_0, 2\frac{-bc - ad}{a^2 + b^2} - q_0\right).$$

Recalling some theorems of earlier chapters, we see that P_i is the midpoint of $\overline{P_0P_2}$, hence that the reflection of P_0 in P_i is P_2. Thus for any P in S its reflection in P_i is the reflection in L_2 of the reflection of P in L_1, and is therefore in S.

18. Let $A = \{(x,y) : \text{for some } t\ ,\ x = at + b \text{ and } y = f(t)\}$; the graph of the parametric equations $x = at + b$, $y = f(t)$.

$B = \{(x,y) : y = f(\frac{x - b}{a})\}$; the graph of $y = f(\frac{x - b}{a})$.

We wish to show $A = B$. Assume $a \neq 0$. Suppose $(p,q) \in A$. This is so if and only if there is a t such that

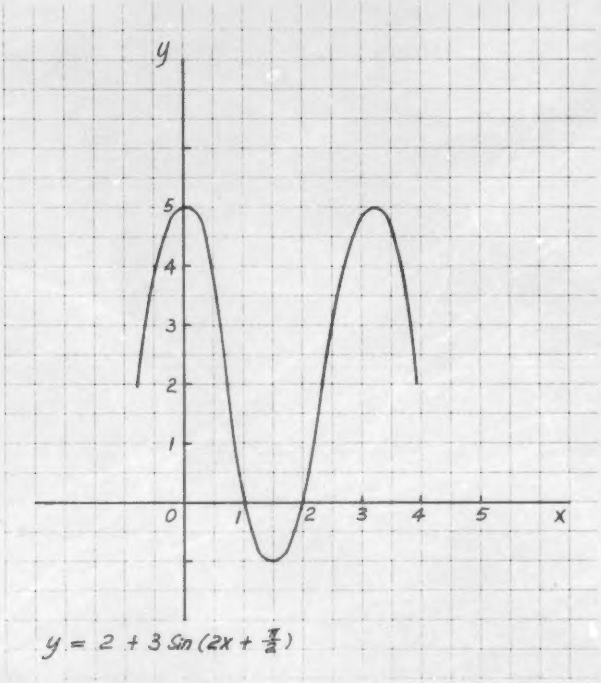
$$p = at + b \quad \text{and} \quad q = f(t)$$

which is so if and only if

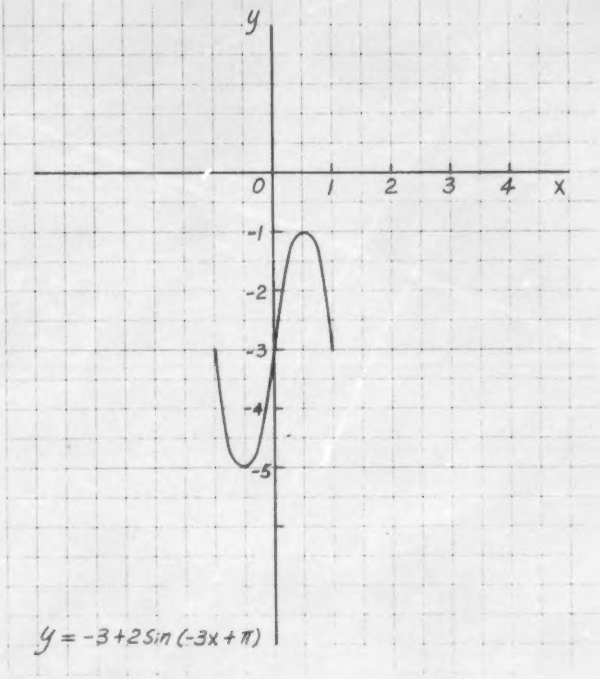
$$t = \frac{p - b}{a} \quad \text{and} \quad q = f(\frac{p - b}{a})$$

which is the defining condition that $(p,q) \in B$. Thus (p,q) A if and only if $(p,q) \in B$, i.e., $A = B$.

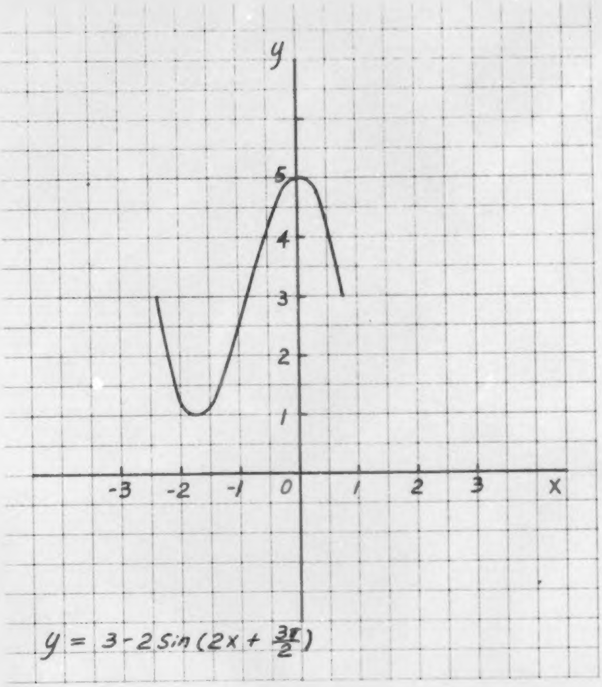
19. (a)



(b)

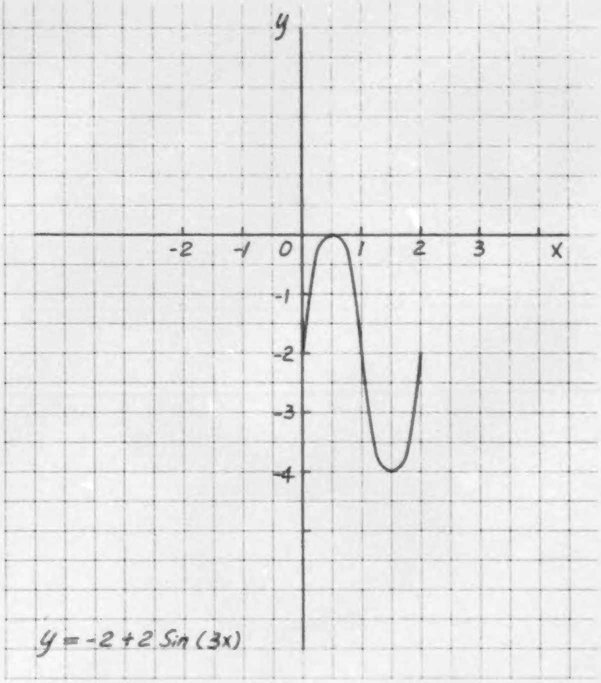


(c)



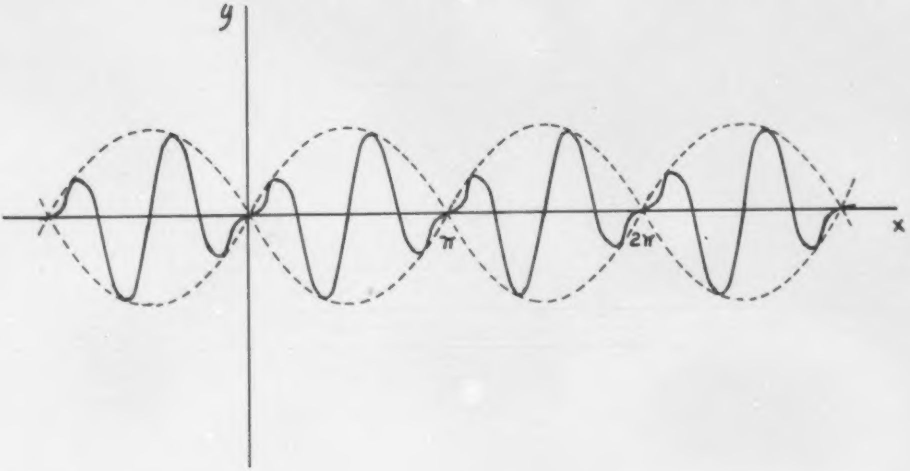
$y = 3 - 2\sin(2x + \frac{3\pi}{2})$

(d)



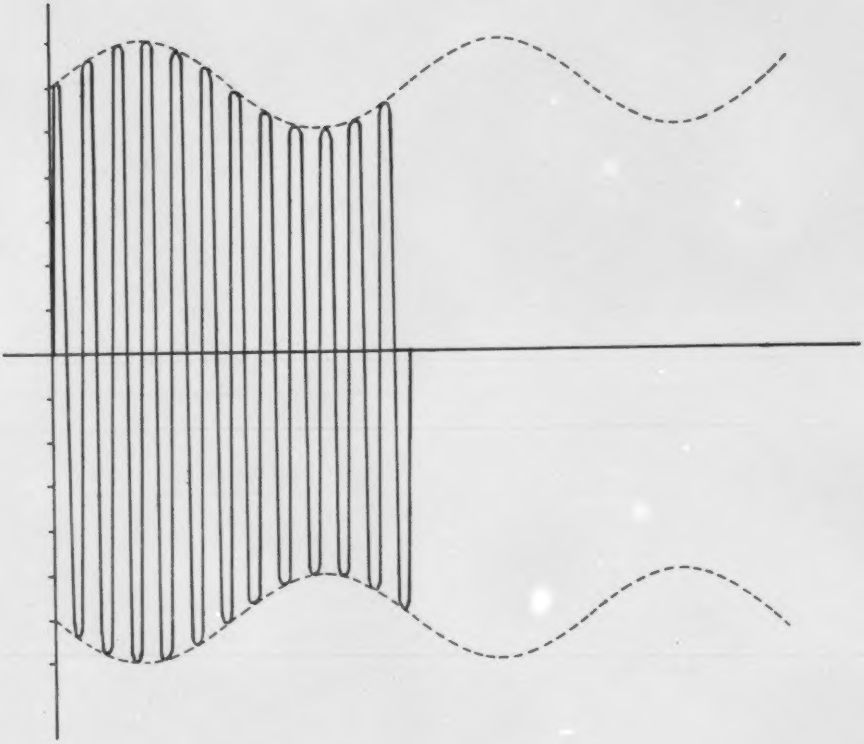
CHALLENGE EXERCISES

1. The graph of $y = \sin 4x \sin x$ may be thought of as a "rapidly" oscillating sine curve. $y_1 = \sin 4x$, modulated by a "slower" oscillating sine curve, $y_2 = \sin x$. Then, as in Example 8, the graph of $y = y_1 y_2$ will be constrained between the graphs of $y_2 = \sin x$ and $y_3 = \sin x$. The graph of y will touch the graph of y_2 whenever $y_1 = \pm 1$, that is, when $x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \dots$. The graph of y will cross the x axis when either y_1 or y_2 equals zero, that is, at $0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \dots$. The graph looks like this (different scales on the axes) :



81

The graph of $y = (6 + \sin x) \sin 12x$ is also a rapidly oscillating curve $y_2 = \sin 12x$ modulated by a slower oscillating curve $y_1 = 6 + \sin x$. The graph of y_1 is a sine curve elevated 6 units above the x-axis. It is bound between 5 and 7, therefore the graph of $y = y_1 y_2$ is constrained by symmetric curves $y_1 = 6 + \sin x$ and $y_3 = -6 - \sin x$ which bound a horizontal strip of periodically varying width, narrowest from 5 to -5 and widest from 7 to -7. The rest of the analysis of the graph is similar to that of the previous paragraph. The graph is drawn below (different scales along the axes).



The graph of $y = \sin(1000\pi t)\sin(1000\,000\pi t)$ will not be drawn but it may be analyzed in the same way as the others. We have a rapidly oscillating curve, $y_2 = \sin(1000000\pi t)$ modulated by a slower oscillating curve $y_1 = \sin(1000\pi t)$. Physicists would say that the first curve has a frequency of 500,000 cycles or 500 kilocycles or .5 megacycles per second. This is a reasonable radio frequency (RF). The second curve then has a frequency of 500 cps which is a reasonable audio frequency (AF). A further discussion of cycles and frequency would lead us too far into physics, and is left for further investigation there.

Teachers' Commentary

Chapter 7

CONIC SECTIONS

The student of Intermediate Mathematics has studied conic sections with equations given in simple form in rectangular coordinates; here we begin with something different. After taking up the introductory material in Section 7-1 and 7-2, if you feel that there is time, you may want to take next the first five sections of the Supplement to Chapter 7. In this you will find a careful development of the geometry of the plane sections of a right circular cone. This development relates the geometric properties of the conics to the cone, the cutting plane, and the sphere tangent to both of them. It is shown that, for a given conic section, the ratio of two cosines is a constant; this ratio, of course, is the eccentricity.

Section 7-3 develops equations in polar form for conics with focus at the pole, first with directrix perpendicular to the polar axis, and then with directrix parallel to the polar axis. (Cases in which there is rotation about the pole will be considered in Chapter 10.) The polar form emphasizes the essential similarity of the locus conditions for the ellipse, hyperbola, and parabola. Transformation of polar equations to familiar forms of the equations of the conics in rectangular coordinates is dealt with in Section 7-4. The exercises in both of these sections provide desirable review of polar coordinates.

The four positions of a conic considered in the text of Section 7-3, in Example 2, and in Exercises 9 and 10, give four forms of the equation which we summarize here.

$$r = \frac{ep}{1 \pm e \cos \theta} \qquad \begin{cases} \text{directrix } \perp \text{ polar axis} \\ + \text{ if directrix contains } (p,0) \\ - \text{ if directrix contains } (-p,0) \end{cases}$$

$$r = \frac{ep}{1 \pm \sin \theta} \qquad \begin{cases} \text{directrix } \parallel \text{ polar axis} \\ + \text{ if directrix contains } (p,\frac{\pi}{2}) \\ - \text{ if directrix contains } (-p,\frac{\pi}{2}) \end{cases}$$

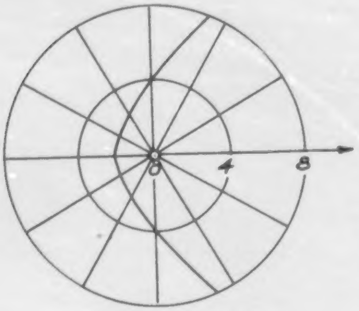
In all these cases the focus is at the pole.

The students should be urged, in doing Exercises 7-3, not to use point-by-point plotting alone. If they first rewrite the equations in a standard form, as indicated in Examples 3 and 4, they can tell what kind of conic section is represented. Then they should find intercepts, a few more points, and use symmetry.

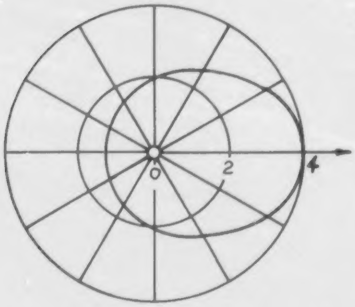
For the convenience of the teacher in making assignments, most of the exercises in this chapter are arranged so that even and odd exercises are roughly comparable. This does not include the applications toward the end of certain sets, or the challenge problems. In the case of exercises such as and 2 of Exercises 7-6, or 1 and 2 of the Review Exercises, with long lists of lettered parts, (a) , (c) , (e) , ... are comparable to (b), (d) , (f) , ...

Exercises 7-3

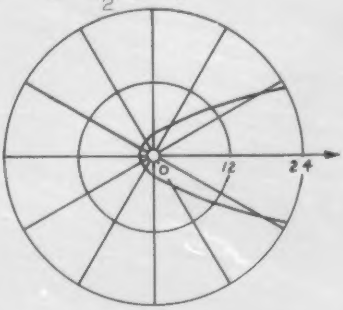
1. $r = \dfrac{4}{1 - \cos\theta}$



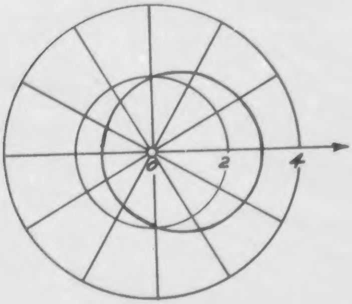
2. $r = \dfrac{3}{1 - \cos\theta}$



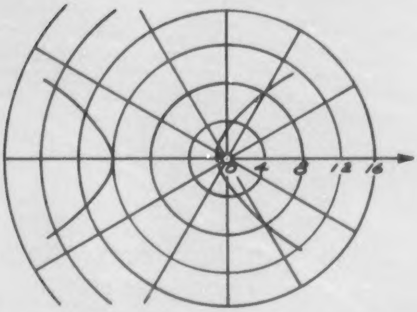
3. $r = \dfrac{2}{1 - \frac{1}{2}\cos\theta}$



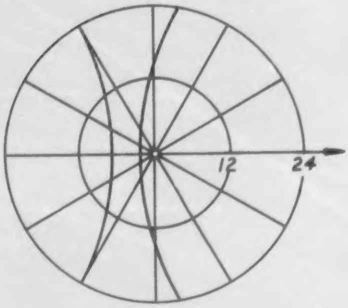
4. $r = \dfrac{2}{1 - \frac{1}{3}\cos\theta}$



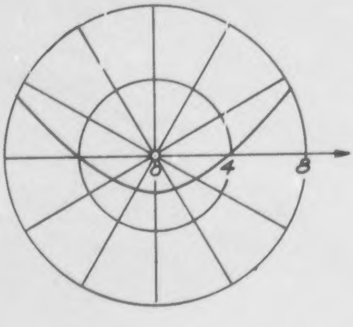
5. $r = \dfrac{3}{1 - \frac{5}{4}\cos\theta}$



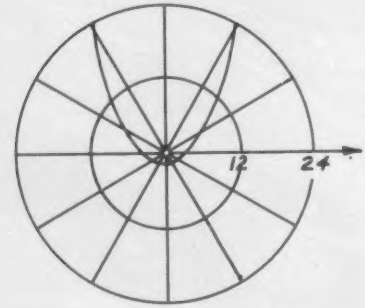
6. $r = \dfrac{12}{1 - 3\cos\theta}$



7. $r = \dfrac{4}{1 - \sin\theta}$



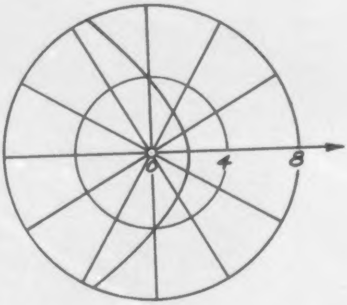
8. $r = \dfrac{3}{1 - \sin\theta}$



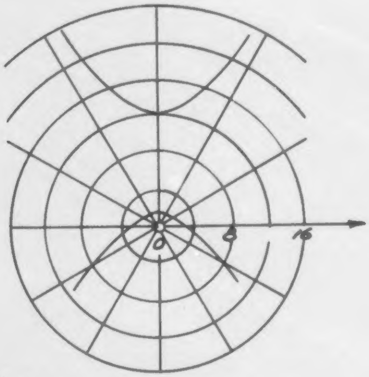
9. $r = \dfrac{ep}{1 + e\sin\theta}$

10. $r = \dfrac{ep}{1 + e\cos\theta}$

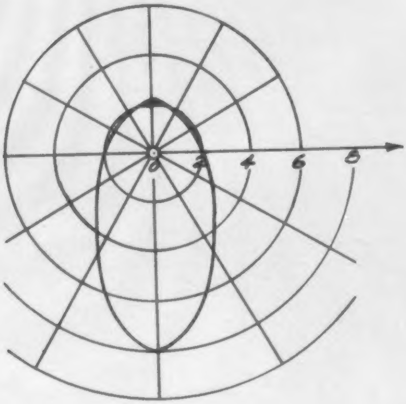
11. (a) $r = \dfrac{4}{1 + \cos\theta}$



(b) $r = \dfrac{3}{1 + \frac{5}{4}\sin\theta}$

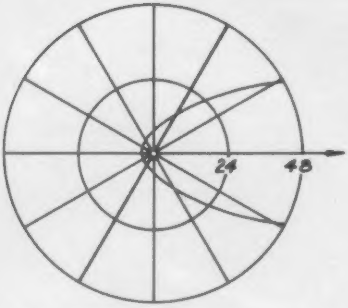


(c) $r = \dfrac{2}{1 + \frac{3}{4}\sin\theta}$

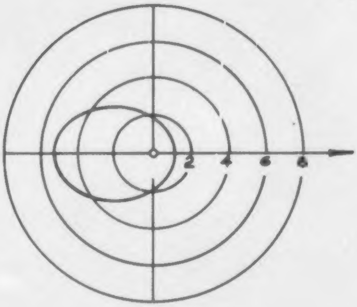


12. $r = \dfrac{6}{1 - \cos\theta}$

Parabola

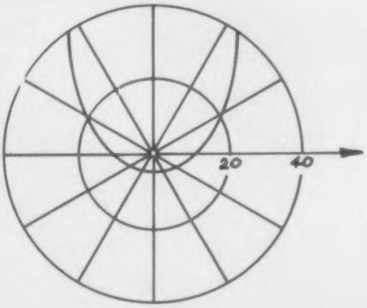


(d) $r = \dfrac{2}{1 + \frac{3}{5}\cos\theta}$



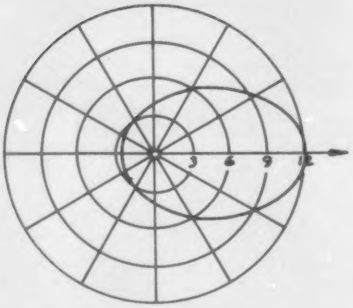
13. $r = \dfrac{10}{1 - \sin\theta}$

Parabola



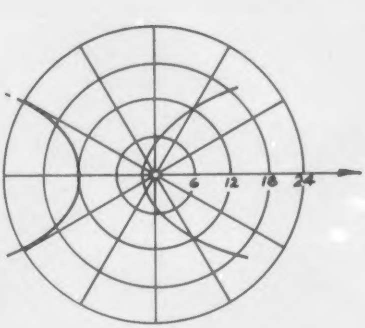
14. $r = \dfrac{4}{1 - \frac{2}{3}\cos\theta}$

Ellipse



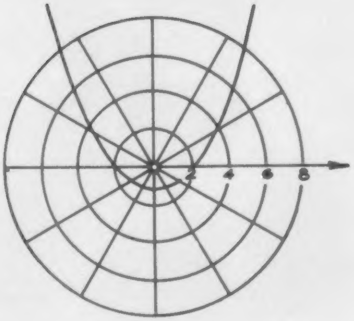
15. $r = \dfrac{4}{1 - \frac{4}{3}\cos\theta}$

Hyperbola



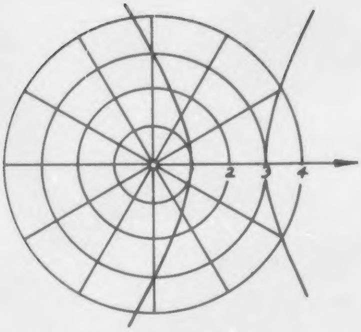
16. $r = \dfrac{2}{1 - \sin\theta}$

Parabola



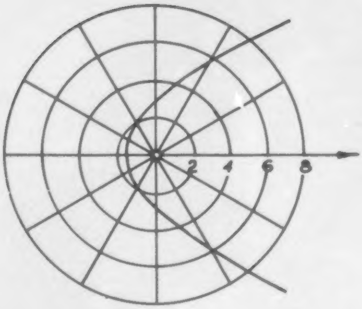
17. $r = \dfrac{3}{1 + 2\cos\theta}$

Hyperbola



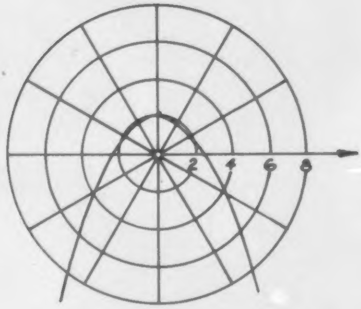
18. $r = \frac{3}{1 - \cos\theta}$

Parabola



19. $r = \frac{-2}{1 - \sin\theta}$

Parabola



20. The path is an ellipse with equation $r = \frac{ep}{1 - e\cos\theta}$. Two points of the path are given: $(5000, 180^\circ)$ and $(6000, 90^\circ)$. We substitute these coordinates in the equation and solve the resulting equations simultaneously to obtain $e = .2$, $p = 30{,}000$. The least distance is 5000 mi.; the greatest, when $\theta = 0^\circ$, is 7500 mi.

In Section 7-4 emphasis is on the algebra involved in transforming from the polar equations of the conic sections to the corresponding rectangular form. In the text we square both members of an equation to obtain Equation (2). It is important that students understand why this is permissible; no doubt they have been warned that so doing may introduce points not in the graph of the original equation. The justification in the text depends on showing that squaring, in effect, introduces a new equation which is the related polar equation of the original equation and hence has the same graph. You might prefer a different proof, somewhat as follows.

Parabola: Since $x = r \cos \theta$, $\cos \theta = \frac{x}{r}$.

Then
$$r = \frac{p}{1 - \cos \theta}$$

becomes
$$r = \frac{p}{1 - \frac{x}{r}}$$

$$= \frac{rp}{r - x} .$$

This is the case if
$$r - x = p$$

or
$$r = x + p .$$

(It would appear that we have divided through by r at this point. This would mean that we would lose the solution $r = 0$. However, since $\cos \theta = \frac{x}{r}$, and consequently is not defined for $r = 0$, this value was not included in the original equation. In the following step we are in effect multiplying by an alternative equation with the same graph.)

We square both sides to obtain

$$r^2 = x^2 + 2px + p^2 ,$$

and substitute for r^2 to obtain

$$x^2 + y^2 = x^2 + 2px + p^2$$

or

$$y^2 = 2p(x + \frac{p}{2}) ,$$

which is a recognizable form for a parabola with vertex at $(-\frac{p}{2}, 0)$.

Ellipse: If

$$r = \frac{ep}{1 - e \cos \theta}, \text{ where } 0 < e < 1 ,$$

then
$$r = \frac{ep}{1 - \frac{ex}{r}}$$

or
$$r = \frac{erp}{r - ex} .$$

90

This is the case if

$$r - ex = ep$$

or $$r = e(x + p) \ .$$

(Once again we have removed the solution $r = 0$ which did not satisfy the original equation. In the following step we are again multiplying by an alternative equation with the same graph.)

We square both sides to obtain

$$r^2 = e^2(x^2 + 2px + p^2)$$

and substitute for r^2 to obtain

$$x^2 + y^2 = e^2(x^2 + 2px + p^2)$$

This is Equation (4) in the text.

We call your attention to the way in which directions for Exercise 7-4 have been written. Depending on what kind of practice your class needs, you would assign all of parts (a), (b), and (c), or just the parts you wish to emphasize.

Exercise 7-4

The graphs are routine and will not be drawn.

1. $x^2 + y^2 = 9$, $r = -3$.
2. $x^2 + y^2 = 81$, $r = -9$.
3. $(x - 1)^2 + y^2 = 1$, related equation the same.
4. $x^2 + y^2 - x - y = 0$, related equation the same.
5. $y^2 = 16 + 8x$, $r = \dfrac{-4}{1 + \cos\theta}$
6. $y^2 = 9 - 6x$, $r = \dfrac{-3}{1 - \cos\theta}$
7. $\dfrac{(x + 2)^2}{1} - \dfrac{y^2}{3} = 3$, $r = \dfrac{-3}{1 + 2\cos\theta}$
8. $\dfrac{(x - 2)^2}{16} + \dfrac{y^2}{12} = 1$, $r = \dfrac{-6}{2 + \cos\theta}$

9. $\frac{(x-2)^2}{9} + \frac{y^2}{5} = 1$, $r = \frac{-5}{3 + 2\cos\theta}$

10. $\frac{(x+3)^2}{4} - \frac{y^2}{5} = 1$, $r = \frac{-5}{2 + 3\cos\theta}$

11. $(x^2 + y^2 - x)^2 = x^2 + y^2$ cardiod curve, $r = \cos\theta - 1$

12. $x^2 = 4 - 4y$, $r = \frac{-2}{1 + \sin\theta}$

13. $7x^2 + 16y^2 - 72x - 144 = 0$, $r = \frac{-12}{4 + 3\cos\theta}$

14. $9y^2 - 16x^2 - 200y + 400 = 0$, $r = \frac{-20}{4 - 5\sin\theta}$

Students of SMSG Intermediate Mathematics will have covered most of the material in Sections 7-5 through 7-8; it is in this text for convenience of reference. Ease in handling the simple forms of the equations of the conics is an important skill.

With able and well-prepared students, only a brief review of the text of these sections will be necessary. However, a number of the exercises should be done, both to reinforce previous learning and to develop further some of the properties and applications of conic sections. With such groups the teacher may want to take up the sections of the Supplement to Chapter 7 which deal with the general second-degree equation.

If the students are not familiar with the equations and basic properties of the conic sections summarized in the first paragraphs of these sections, the teacher will want to take time to develop this material with the class. Intermediate Mathematics would be helpful here.

While the amount of time that should be devoted to these sections will vary greatly with the training and ability of the class, it is urged that the time be sufficient for the students to develop some facility both with use of the locus definitions and with the standard forms.

The answers for Exercises 7-5 through 7-8 do not, in most cases, include the sketches the students are asked to make. However, use of the listed information about the curves will make it easy to check the students' sketches.

Exercises 7-5

1. Equation	Vertex	Focus	Axis	Directrix
(a) $x^2 = 4(-4)y$	$(0,0)$	$(0,-4)$	$x = 0$	$y = 4$
(b) $y^2 = 4(4)x$	$(0,0)$	$(4,0)$	$y = 0$	$x = -4$
(c) $x^2 = 4(\frac{3}{20})y$	$(0,0)$	$(0,\frac{3}{20})$	$x = 0$	$y = -\frac{3}{20}$
(d) $(y - \frac{5}{2})^2 = 4(-\frac{3}{2})(x - \frac{89}{24})$	$(\frac{89}{24},\frac{5}{2})$	$(\frac{53}{24},\frac{5}{2})$	$y = \frac{5}{2}$	$x = \frac{125}{24}$
(e) $(x - 2)^2 = 4(\frac{3}{8})(y - 1)$	$(2,1)$	$(2,\frac{11}{8})$	$x = 2$	$y = \frac{5}{8}$
(f) $(x+\frac{b}{2a})^2=4(\frac{1}{4a})(y-\frac{4ac-b^2}{4a})$	$(-\frac{b}{2a},\frac{4ac-b^2}{4a})$	$(-\frac{b}{2a},\frac{4ac-b^2+1}{4a})$	$x=-\frac{b}{2a}$	$y=\frac{4ac-b^2-1}{4a}$

2. (a) Case (a). Equation $x^2 = 0$. **The y-axis.**

Case (b). Equation: $(x - h)^2 = 0$. **A line parallel to the y-axis.**

Case (c). Equation: $y^2 = 0$. **The x-axis.**

(b) These cases occur when the cutting plane contains an element of the cone.

3. (a) $(y + 2)^2 = -6(x - \frac{1}{2})$

(b) $(x + 1)^2 = 2(y - \frac{5}{2})$

(c) $y^2 = -20x$

(d) $(y - 5)^2 = 4(x - 4)$

4. Same answers as for Problem 3.

5. (a) $y^2 = -10x$

(b) $(x - 2)^2 = -12(y + 3)$

(c) $y^2 = \frac{49}{2}x$

(d) $(x + 2)^2 = -16(y - 3)$

6. The equally spaced rulings permit locating points that are equally distant from a fixed line (L_0) and a fixed point (F). Thus P_2 is two units from L_0 (since it is on the second ruling away from L_0) and it is two units from F (since the radius used to determine it was two units, with F as center).

7. For every position of the pencil point P, the distance from P to the fixed line (L) is equal to the distance from P to the fixed point F.

Challenge Problems

1. The focus of the parabola is $F = (0, \frac{1}{4})$: the slope of the line containing P and F is $\frac{4a^2 - 1}{4a}$. Using this and the slope of the tangent line $(2a)$, we find that the tangent of the angle these lines form is $\frac{1}{2a}$. To avoid the problem presented by a vertical line, we use the fact that the angle between the tangent line and the parallel to the axis of the parabola is the complement of the angle formed by the tangent line and the x-axis; tangent of this last angle is $2a$; hence the tangent of its complement is $\frac{1}{2a}$.

2. (a) The tangent perpendicular to the line $y = 2ax - a^2$ must have slope $-\frac{1}{2a}$; therefore its point of contact is $P^1 = (-\frac{1}{4a}, \frac{1}{16a^2})$

   A test for collinearity can then be applied to the coordinates of P, P', and V.

   (b) Using the previous results, we obtain the equation of the tangent at P': $8ax + 16a^2y + 1 = 0$. We apply a test of concurrency to this equation, the equation of the tangent at P, and the equation of the directrix $(4y + 1 = 0)$.

Example 2 of Section 7-6 will give an opportunity to review with the students the technique of completing the square. Here the coefficients are numerical; when the method was used in Section 7-4 the coefficients were literal. Since the technique will continue to be useful here and elsewhere, we recommend that the teacher check that the students have facility with it. They should be able to handle not just the simplest cases (like the first ones in the exercise set), but also ones like 1(g) , (h) and 5(c) of this set, and 3(g) , (h) of Exercises 7-7.

Exercises 7-6

1. (a) $(x - 4)^2 + y^2 = 16$ $C = (4,0)$ $r = 4$
   (b) $(x - 3)^2 + (y - 5)^2 = 1$ $C = (3,5)$ $r = 1$
   (c) $(x - 2)^2 + (y + 4)^2 = 0$ Locus is the point $(2,-4)$.
   (d) $(x + 7)^2 + (y - \frac{9}{2})^2 = \frac{37}{4}$ $C = (-7,\frac{9}{2})$ $r = \frac{1}{2}\sqrt{37}$
   (e) $(x - \frac{1}{2})^2 + (y + \frac{1}{2})^2 = \frac{1}{2}$ $C = (\frac{1}{2},-\frac{1}{2})$ $r = \frac{1}{2}\sqrt{2}$
   (f) $(x - a)^2 + (y - b)^2 = 0$ Locus is the point (a,b) .
   (g) $(x - \frac{3}{5})^2 + (y + \frac{2}{5})^2 = \frac{3}{25}$ $C = (\frac{3}{5},-\frac{2}{5})$ $r = \frac{1}{5}\sqrt{3}$
   (h) $(x - \frac{a}{2})^2 + (y + \frac{b}{2})^2 = \frac{a^2 + 2ab + b^2}{4}$ $C = (\frac{a}{2},-\frac{b}{2})$ $r = \frac{a + b}{2}$

2. (a) $x^2 + y^2 - 6x + 10y - 15 = 0$
   (b) $x^2 + y^2 + 10x - 24y = 0$
   (c) $x^2 + y^2 - 6x - 4y + 9 = 0$ and $x^2 + y^2 - 6x - 4y + 4 = 0$
   (d) The center is $(2,1)$ or $(-1,4)$.
   Equations: $x^2 + y^2 - 4x - 2y - 4 = 0$ and $x^2 + y^2 + 2x - 8y + 8 = 0$.
   (e) $r = \frac{17}{5}$. Equation: $25x^2 + 25y^2 - 50x - 100y - 164 = 0$
   (f) If equation is written $x^2 + y^2 + Dx + Ey + F = 0$, substitution of the coordinates gives equations $2D + 3E + F + 13 = 0$,
   $5D + E + F + 26 = 0$, $3E + F + 3 = 0$. Final equation:
   $x^2 + y^2 - 5x - y = 0$

3. (a) Slope of radius to $(3,-4)$ is $-\frac{4}{3}$; therefore slope of tangent is $\frac{3}{4}$. Equation: $3x - 4y - 25 = 0$

   (b) Proceding as in part (a), equation of tangent is $x_1x + y_1y = x_1^2 + y_1^2$. Since (x_1,y_1) is a point of the circle, $x_1^2 + y_1^2 = r^2$; thus the desired equation is $x_1x + y_1y = r^2$.

4. (a) Since the center $(0,0)$, the point $(3,7)$, and a point of contact of the tangent determines a right triangle, the Pythagorean Theorem can be used. Length of tangent $= \sqrt{58 - 25} = \sqrt{33}$.

   (b) [See part (a)] $C(-\frac{D}{2}, -\frac{E}{2})$, $r = \frac{1}{2}\sqrt{D^2 + E^2 - 4F}$

$$t^2 = (x_1 + \frac{D}{2})^2 + (y_1 + \frac{E}{2})^2 - \frac{1}{4}(D^2 + E^2 - 4F)$$

$$= x_1^2 + y_1^2 + Dx_1 + Ey_1 + F$$

   (c) If $t^2 = 0$, the point (x_1,y_1) is a point of the circle. If $t^2 < 0$, the distance from the center to (x_1,y_1) is less than the radius; hence the point (x_1,y_1) is a point of the interior of the circle.

5. (a) $a(x^2 + y^2 - 10x - 2y - 35) + b(x^2 + y^2 + 4x - 6y - 49) = 0$

   (b) Substitution of $x = 0$, $y = -6$ in (a) gives $13a = -23b$. If we let $a = 23$ and $b = 13$, an equation is

$$5x^2 + 5y^2 - 141x + 16y - 84 = 0.$$

   (c) The terms containing x in the equation of part (a) are

$$(a + b)x^2 + (-10a + 4b)x.$$

Thus the x-coordinate of the center (which we know is -5) is $\frac{5a - 2b}{a + b}$. From this we find $10a = -3b$; we let $a = -3$, $b = 10$. Using these values, the equation is $7x^2 + 7y^2 + 70x - 54y - 385 = 0$.

6. Let the circles have equations $(x - h)^2 + (y - k)^2 = r^2$ and $(x - h_1)^2 + (y - k_1)^2 = r_1^2$. Then an equation of the radical axis is $(x - h)^2 + (y - k)^2 - r^2 - \left((x - h_1)^2 + (y - k_1)^2 - r_1^2\right) = 0$.
For either circle, the square of the length of the tangent from the point (x,y) to the circle is the square of the distance from the point to the center, minus the square of the radius. The condition that these two lengths be equal is

$$(x - h)^2 + (y - k)^2 - r^2 = (x - h_1)^2 + (y - k_1)^2 - r_1^2$$

But this is exactly the condition shown above, that the point (x,y) is on the radical axis.

7. As shown in Problem 6, the required point must be on the radical axis of each pair of circles. We find equations of two of the radical axes (say $6x - y = -8$ and $4x - 3y = 11$) and solve; the point is $(-\frac{5}{2}, -7)$.

8. Using the circles with equations in Problem 6, slope of line of centers is $\frac{k_1 - k}{h_1 - h}$. From equation of radical axis (also Problem 6), slope is $-\frac{h_1 - h}{k_1 - k}$, the negative reciprocal of slope of line of centers. (If $h_1 = h$, first line is parallel to y-axis and second to x-axis, and also they are perpendicular; opposite case if $k_1 = k$.)

9. For the first circle, we have $C_1 = (-\frac{D_1}{2}, -\frac{E_1}{2})$, $r_1^2 = \frac{D_1^2 + E_1^2 - 4F_1}{4}$; for the second, $C_2 = (-\frac{D_2}{2}, -\frac{E_2}{2})$, $r_2^2 = \frac{D_2^2 + E_2^2 - 4F_2}{4}$. By the definition of orthogonal circles, $r_1^2 + r_2^2 = d(C_1, C_2)^2$; this condition is $\frac{D_1^2 + E_1^2 - 4F_1}{4} + \frac{D_2^2 + E_2^2 - 4F_2}{4} = (-\frac{D_1}{2} + \frac{D_2}{2})^2 + (-\frac{E_1}{2} + \frac{E_2}{2})^2$.
When simplified, this is the desired condition.

10. Use the condition in Problem 11; in (b) both members of the equations must be divided by 2 before the condition applies.

11. (a) $k = -2$
(b) $k = 48$ (Equations must be rewritten in proper form.)

Challenge Problems

1. Let the equations of the circles be

$$C_1 : x^2 + y^2 + D_1x + E_1y + F_1 = 0$$

$$C_2 : x^2 + y^2 + D_2x + E_2y + F_2 = 0$$

$$C_3 : x^2 + y^2 + D_3x + E_3y + F_3 = 0 .$$

Then equations of the common chords of C_1 and C_2 , C_2 and C_3 , and C_1 and C_3 are, respectively,

$$L_1 : (D_1 - D_2)x + (E_1 - E_2)y + F_1 - F_2 = 0$$

$$L_2 : (D_2 - D_3)x + (E_2 - E_3)y + F_2 - F_3 = 0$$

$$L_3 : (D_1 - D_3)x + (E_1 - E_3)y + F_1 - F_3 = 0$$

The family of lines through the intersection of L_1 and L_2 has as an equation

$$a\big((D_1 - D_2)x + (E_1 - E_2)y + F_1 - F_2\big) + b\big((D_2 - D_3)x + (E_2 - E_3)y + F_2 - F_3\big) = 0$$

For the values $a = 1$ and $b = 1$ this equation becomes

$$(D_1 - D_3)x + (E_1 - E_3)y + F_1 - F_3 = 0.$$

But this is exactly the equation we had for L_3 ; hence the lines are concurrent.

(Note: This is, of course, not the only way to make this proof. It is possible to assign coordinates to the vertices of the triangle, and find, in terms of these coordinates equations of the common chords and coordinates of their point of intersection.)

2. The proof given here for Challenge Problem 1 also holds here; so would any other that did not use the fact that in Problem 1 the circles intersect,

The student is asked to explain the variation in shape of the ellipse from the fact that $b = a\sqrt{1 - e^2}$. He should be able to see that the nearer the value of e is to zero, the closer $\sqrt{1 - e^2}$ is to 1 ; in such cases the minor axis differs very little from the major axis. But if values close to 1 (but less than 1) are selected for e , $\sqrt{1 - e^2}$ can be made as small as one wishes, and hence the minor axis can be made small as compared with the major axis.

Exercises 7-7

1. $\frac{(x - 3)^2}{36} + \frac{(y - 2)^2}{16} = 1$; $F(3 + 2\sqrt{5},2)$, $F'(3 - 2\sqrt{5},2)$, $V(9,2)$, $V'(-3,2)$; $x = 3 \pm \frac{18}{5}\sqrt{5}$; $e = \frac{\sqrt{5}}{3}$

2. $\frac{x^2}{9} + \frac{y^2}{5} = 1$

3.

	Equation	e	F,F'	V,V'	Directrices
(a)	$\frac{x^2}{1^2} + \frac{y^2}{2^2} = 1$	$\frac{\sqrt{3}}{2}$	$(0,\pm\sqrt{3})$	$(0,2),(0,-2)$	$y = \pm\frac{4}{3}\sqrt{3}$
(b)	$\frac{x^2}{5^2} + \frac{y^2}{2^2} = 1$	$\frac{\sqrt{21}}{5}$	$(\pm\sqrt{21},0)$	$(\pm 5,0)$	$x = \pm\frac{25}{21}\sqrt{21}$
(c)	$\frac{x^2}{(\sqrt{2})^2} + \frac{y^2}{(\sqrt{3})^2} = 1$	$\frac{\sqrt{3}}{3}$	$(0,\pm 1)$	$(0,\pm\sqrt{3})$	$y = \pm 3$
(d)	$\frac{x^2}{(\frac{1}{2})^2} + \frac{y^2}{(\frac{1}{3})^2} = 1$	$\frac{\sqrt{5}}{3}$	$(\pm\frac{\sqrt{5}}{6},0)$	$(\pm\frac{1}{2},0)$	$x = \pm\frac{3}{10}\sqrt{5}$
(e)	$\frac{(x - 4)^2}{5^2} + \frac{(y + 3)^2}{6^2} = 1$	$\frac{\sqrt{11}}{6}$	$(4,-3 \pm \sqrt{11})$	$(4,3),(4,-9)$	$y = -3 \pm \frac{36}{11}\sqrt{11}$
(f)	$\frac{(x + 5)^2}{3^2} + \frac{(y + 1)^2}{2^2} = 1$	$\frac{\sqrt{5}}{3}$	$(-5 \pm \sqrt{5},-1)$	$(-2,-1),(-8,-1)$	$x = -5 \pm \frac{9}{5}\sqrt{5}$
(g)	$\frac{(x - 2)^2}{2^2} + \frac{y^2}{3^2} = 1$	$\frac{\sqrt{5}}{3}$	$(2,\pm\sqrt{5})$	$(2,3),(2,-3)$	$y = \pm\frac{9}{5}\sqrt{5}$
(h)	$\frac{(x + 1)^2}{2^2} + \frac{(y - 5)^2}{4^2} = 1$	$\frac{\sqrt{3}}{2}$	$(-1,5 \pm 2\sqrt{3})$	$(-1,9),(-1,1)$	$y = 5 \pm \frac{8}{3}\sqrt{3}$
(i)	$\frac{(x - 1)^2}{5^2} + \frac{(y + 3)^2}{4^2} = 0$	Locus is the point $(1,-3)$.			

4. (a) $\frac{x^2}{16} + \frac{y^2}{9} = 1$

   (b) $\frac{x^2}{5} + \frac{y^2}{9} = 1$

   (c) $\frac{(x - 3)^2}{25} + \frac{(y - 5)^2}{\frac{600}{49}} = 1$

   (d) $\frac{(x - 1)^2}{16} + \frac{(y - 4)^2}{12} = 1$

5. The latus rectum of an ellipse is either of the two chords of an ellipse perpendicular to the major axis at a focus.

   If in Equation (a) of Figure 7-4, we set $x = ae$, we find $y = \pm b\sqrt{1 - e^2} = \pm \frac{b^2}{a}$; thus, the length of a latus rectum is $\frac{2b^2}{a}$. (The same result is obtained in each of the other forms.)

6. For the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$,

$$\begin{aligned} r + r' &= \sqrt{(x - ae)^2 + y^2} + \sqrt{(x + ae)^2 + y^2} \\ &= \sqrt{(x - ae)^2 + (1 - e^2)(a^2 - x^2)} + \sqrt{(x + ae)^2 + (1 - e^2)(a^2 - x^2)} \\ &= \sqrt{a^2 - 2aex + e^2x^2} + \sqrt{a^2 + 2aex + e^2x^2} \\ &= a - ex + a + ex \\ &= 2a \end{aligned}$$

   It should be noted that the first radical expression is equal to $a - ex$ rather than $ex - a$ because the largest possible x is a, and e is less than one; hence $a - ex$ is positive.

7. If $P(x,y)$ is any point on the ellipse, the fixed points are $F(c,0)$, $F'(-c,0)$, and the constant is $2a (a > c)$, then

   (1) $$PF + PF' = 2a$$

$$\sqrt{(x - c)^2 + y^2} + \sqrt{(x + c)^2 + y^2} = 2a .$$

   After eliminating radicals in the usual way, this becomes

$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$$

or, we let

$$b^2 = a^2 - c^2 ,$$

$$b^2x^2 + a^2y^2 = a^2b^2$$

(2) or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 .$$

This sketches the proof that if the coordinates of a point satisfy (1), they satisfy (2). For the converse, we retrace our steps, but must use both signs when the square root is taken, so that there are four equations,

$$\pm\sqrt{(x - c)^2 + y^2} \pm \sqrt{(x + c)^2 + y^2} = 2a .$$

It can easily be shown, because of the requirement that $a > c$ and the fact that the two radicals represent two sides of a triangle of which the third side has length $2c$, that only the positive signs can be used.

8. Each point is located so that the sum of its distances from the fixed points (F,F') is a constant (the length of $\overline{VV'}$, greater than the length of $\overline{FF'}$).

9. See Problem 8. As the distance between the tacks increases, the ellipse becomes more elongated; as it decreases, the ellipse becomes more like a circle.

10. (a) $5x^2 + 9y^2 - 40x - 54y + 116 = 0$

    (b) $8x^2 - 4xy + 5y^2 - 38x - 58y + 242 = 0$

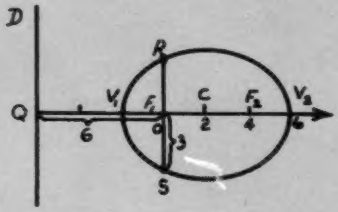
11. $e = 0$; the focus-directrix definition cannot be used for a circle.

12. The ellipse has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and is symmetric with respect to the origin and to both of the coordinate axes. Therefore $(-c,0)$ and the line $x = -\frac{a}{e} = -\frac{c}{e^2}$ are also a focus and directrix of the ellipse. (See Figure 7-4 part (a)).

13. We may recognize $r = \dfrac{6}{2 - \cos\theta}$ as an equation of an ellipse, and rewrite the equation in rectangular form for the purpose of discussion as in Example 1. Instead, we shall carry out the discussion in polar coordinates in order to illustrate the procedure in that system. We may rewrite the equation as

$$r = \frac{\frac{1}{2}(6)}{1 - \frac{1}{2}\cos\theta}$$

and see that the graph is an ellipse with one focus at the pole, with eccentricity $e = \frac{1}{2}$, and with directrix six units to the left of the pole and perpendicular to the line along the polar axis. From the definition of eccentricity we have



$$e = \frac{1}{2} = \frac{d(F_1,V_1)}{d(V_1,Q)} = \frac{d(F_1,V_2)}{d(V_2,Q)} .$$

Since $d(F_1,Q) = 6$, we have $d(F_1,V_1) = 2$, $d(F_1,V_2) = 6$. Therefore the vertices are $V_1 = (2,\pi)$ and $V_2 = (6,0)$. Since $d(F_2,V_2) = d(F_1,V_1) = 2$, we have the coordinates of the other focus, $F_2 = (4,0)$. Since the center of the ellipse is the midpoint of $\overline{F_1F_2}$, we have $C = (2,0)$. We readily find the major axis, $2a = d(V_1,V_2) = 8$, and the focal distance, $2c = d(F_1,F_2) = 4$. (We verify that $e = \frac{c}{a} = \frac{2}{4} = \frac{1}{2}$.) From the relationship $b^2 = a^2 - c^2$, we have

$b^2 = 4^2 - 2^2 = 12$ and $b = 2\sqrt{3}$, which gives the minor axis, $2b = 4\sqrt{3}$. The length of a latus rectum (only one, RS, is drawn in the figure) can be found from the fact that it is twice the polar distance to the point R, for which $\theta = \frac{\pi}{2}$. Substitution in the original equation gives for this distance

$$d(F_1R) = \frac{6}{2 - \cos\frac{\pi}{2}} = 3 ;$$

therefore each latus rectum is of length 6. Using these values, we complete the sketch.

14. Using the representation $\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$, the desired proportion is

$$2ae : 2a = 2a : 2\frac{a}{e} .$$

This can be verified immediately.

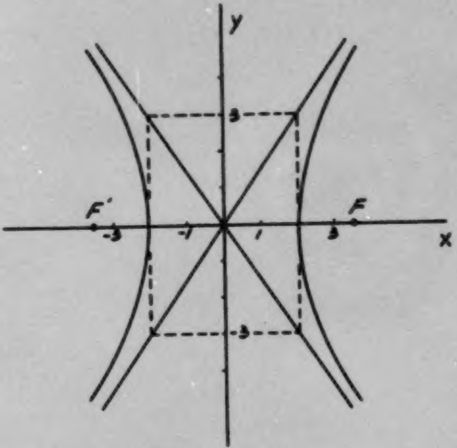
In the first four problems in Exercises 7-8, you will notice that the location of the transverse axis has been specified, but it has not been indicated which of the lengths given for the semi-axes is that of the transverse axis. This is deliberate, and it is suggested that you not make any additional specification in assigning the problems to the students. They should discover for themselves that two different hyperbolas meet the conditions in each problem, and should realize how this case differs from that of the ellipse, where the longer of the two axes must be the major axis.

Exercises 7-8

1. $\frac{x^2}{4} - \frac{y^2}{9} = 1$, $e = \frac{\sqrt{13}}{2}$, $V(2,0)$, $V'(-2,0)$, $F = (\sqrt{13},0)$, $F' = (-\sqrt{13},0)$

$D : x = \frac{4}{13}\sqrt{13}$, $D' : x = -\frac{4}{13}\sqrt{13}$

$A : y = \frac{3}{2}x$, $A' : y = -\frac{3}{2}x$



or

$\frac{x^2}{9} - \frac{y^2}{4} = 1$

$C = \frac{\sqrt{13}}{3}$, $V(3,0)$, $V'(-3,0)$, $F(\sqrt{13},0)$, $F'(-\sqrt{13},0)$

$D : x = \frac{9}{13}\sqrt{13}$, $D' : x = -\frac{9}{13}\sqrt{13}$

$A : y = \frac{2}{3}x$, $A' : y = -\frac{2}{3}x$

2. $-\frac{x^2}{4} + \frac{y^2}{9} = 1$

$e = \frac{\sqrt{13}}{3}$, $V(0,3)$, $V'(0,-3)$, $F(0,\sqrt{13})$, $F'(0,-\sqrt{13})$

$D : y = \frac{9}{13}\sqrt{13}$, $D' : y = -\frac{9}{13}\sqrt{13}$

$A : y = \frac{3}{2}x$, $A' : y = -\frac{3}{2}x$

or

$-\frac{x^2}{9} + \frac{y^2}{4} = 1$

$e = \frac{\sqrt{13}}{2}$, $V(0,2)$, $V'(0,-2)$, $F(0,\sqrt{13})$, $F'(0,-\sqrt{13})$

$D : y = \frac{4}{13}\sqrt{13}$, $D' : y = -\frac{4}{13}\sqrt{13}$

$A : y = \frac{2}{3}x$, $A' : y = -\frac{2}{3}x$

3. $\frac{(x+2)^2}{16} - \frac{(y-3)^2}{9} = 1$

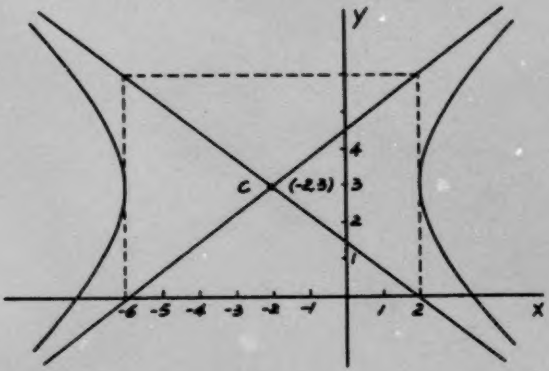
$e = \frac{5}{4}$, $V(2,3)$, $V'(-6,3)$

$F = (3,3)$, $(-7,3)$

$D : x = \frac{6}{5}$, $x = -\frac{26}{5}$

$A : 3x - 4y + 18 = 0$

$A' : 3x + 4y - 6 = 0$



or

$\frac{(x+2)^2}{9} - \frac{(y-3)^2}{16} = 1$

$e = \frac{5}{3}$, $V = (1,3)$, $V'(-5,3)$, $F(3,3)$, $F'(-7,3)$

$D : x = -\frac{1}{5}$, $D' : x = -\frac{19}{5}$

$A : 4x - 3y + 17 = 0$, $A' : 4x + 3y - 1 = 0$

4. $-\frac{(x+2)^2}{16} + \frac{(y-3)^2}{9} = 1$

$e = \frac{5}{3}$, $V(-2,6)$, $V'(-2,0)$, $F(-2,8)$, $F'(-2,-2)$

$D : y = \frac{24}{5}$, $D' : y = \frac{6}{5}$

$A : 3x - 4y + 18 = 0$, $A' : 3x + 4y - 6 = 0$

or

$-\frac{(x+2)^2}{9} + \frac{(y-3)^2}{16} = 1$

$e = \frac{5}{4}$, $V = (-2,7)$, $V'(-2,-1)$, $F = (-2,8)$, $F'(-2,-2)$

$D : y = \frac{31}{5}$, $D' : y = -\frac{1}{5}$

$A : 4x - 3y + 17 = 0$, $A' : 4x + 3y - 1 = 0$

5. See the next page.

6. (a) $-x^2 + y^2 = 4$
   (b) $x^2 - y^2 = 4$
   (c) $-4x^2 + 9y^2 = 36$
   (d) $25x^2 - 144y^2 = 3600$
   (e) $x^2 - 4y^2 - 4x + 24y - 48 = 0$

7. $16x^2 - 9y^2 = 144$

8. $2xy = 1$; $e = \sqrt{2}$

9. It will be easier to do this proof if the coordinate system is chosen in such a fashion that the origin is the midpoint of the line segment determined by the two fixed points.

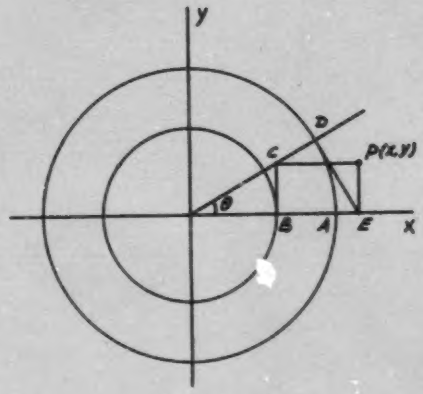
10. The latus rectum of a hyperbola is either of the two focal chords perpendicular to the transverse axis; its length is $\frac{2b^2}{a}$.

11. The points located by the construction lie on a hyperbola because the construction determines each one so that there is a constant difference (2a) in its distances to the two fixed points.

12. Elimination of the parameter gives $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

5.	e	a	b	C	F	F'	V	V'	D,D'	A,A'
(a)	$\sqrt{2}$	2	2	(0,0)	$(2\sqrt{2},0)$	$(-2\sqrt{2},0)$	(2,0)	(-2,0)	$x = \pm\sqrt{2}$	$y = \pm x$
(b)	$\sqrt{2}$	2	2	(0,0)	$(0,2\sqrt{2})$	$(0,-2\sqrt{2})$	(0,2)	(0,-2)	$y = \pm\sqrt{2}$	$y = \pm x$
(c)	$\frac{\sqrt{13}}{3}$	3	2	(0,0)	$(\sqrt{13},0)$	$(-\sqrt{13},0)$	(3,0)	(-3,0)	$x = \pm \frac{9}{13}\sqrt{13}$	$y = \pm \frac{2}{3}x$
(d)	$\frac{13}{5}$	5	12	(0,0)	(0,13)	(0,-13)	(0,5)	(0,-5)	$y = \pm \frac{25}{13}$	$y = \pm \frac{5}{12}x$
(e)	$\sqrt{5}$	2	4	(2,3)	$(2,3 + 2\sqrt{5})$	$(2,3 - 2\sqrt{5})$	(2,5)	(2,1)	$y = 3 \pm \frac{2}{5}\sqrt{5}$	$x - 2y + 4 = 0, x + 2y - 8 = 0$

13. Draw concentric circles, center $(0,0)$, radii a and b, any angle θ. Draw tangent at B (intersecting OD at C) and tangent at D (intersecting x-axis at E). From C and E draw parallels to the x- and y-axes respectively, intersecting at $P(x,y)$. Then $x = OE = a \sec \theta$ and $y = CB = b \tan \theta$.



Hence $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \sec^2 \theta - \tan^2 \theta = 1$.

14. (a) $xy = -21$

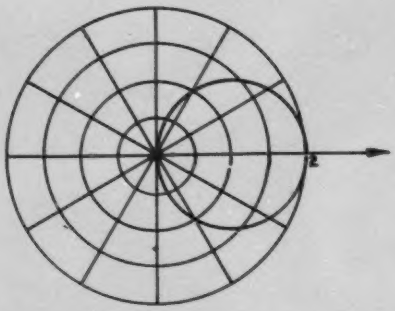
(b) $-x^2 + y^2 = 40$

15. Locus is a pair of intersecting lines (the asymptotes of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$).

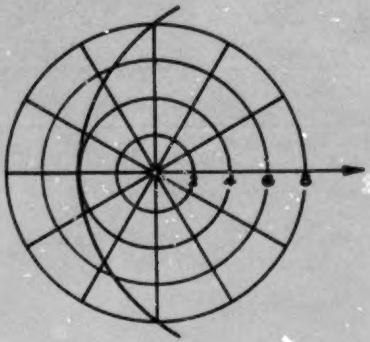
Review Exercises

1. (a) Circle with center at pole and radius of $\frac{2}{3}$.

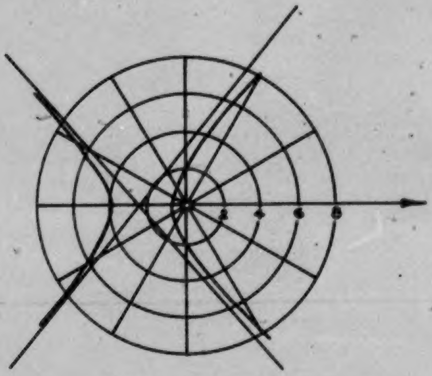
(b) Circle with center at $(1,0^\circ)$ and radius of 1.



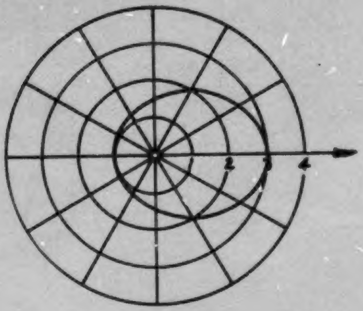
(c) Parabola with vertex at $(4,\pi)$, focus at 0 and directrix perpendicular to polar axis and 8 units to left of pole.



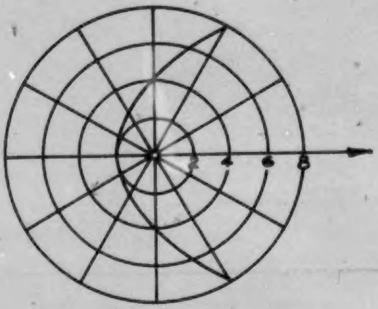
(d) Hyperbola with eccentricity of $\frac{3}{2}$, center at $(-\frac{12}{5}, 0^{\circ})$, foci at $(0,0^{\circ})$ and $(-\frac{24}{5}, 0^{\circ})$, vertices at $(\frac{4}{5}, 180^{\circ})$ and $(-4,0^{\circ})$, and directrices $r \cos \theta = -\frac{4}{3}$ and $r \cos \theta = -\frac{52}{15}$.



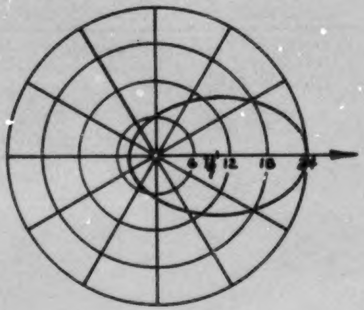
(e) Ellipse with eccentricity of $\frac{1}{2}$, center at $(1,0^\circ)$, foci at $(2,0^\circ)$ and $(0,0^\circ)$, directrices $r \cos \theta = 5$ and $r \cos \theta = -3$, vertices $(3,0^\circ)$ and $(1,180^\circ)$.



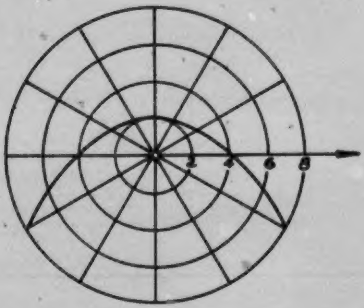
(f) Parabola with eccentricity of 1 , focus at pole, vertex at $(2,180^\circ)$, and directrix $r \cos \theta = -4$.



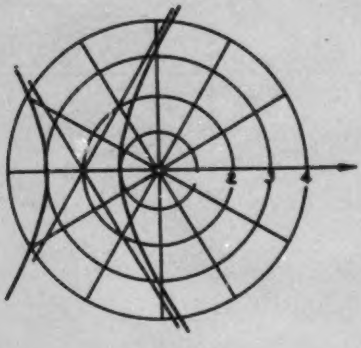
(g) Ellipse with eccentricity $\frac{3}{4}$, center at $(\frac{72}{7}, 0^\circ)$, foci at $(0,0^\circ)$ and $(\frac{144}{7}, 0^\circ)$, vertices at $(24,0^\circ)$ and $(\frac{24}{7}, 180^\circ)$, the length of the minor axis is $\frac{48}{\sqrt{7}}$, and the directrices are $r \cos \theta = \frac{200}{7}$ and $r \cos \theta = -8$.



(h) Parabola with eccentricity 0 , vertex $(2,90^\circ)$, focus at the pole, and directrix the line $r \sin \theta = 4$.



(i) Hyperbola with eccentricity 2, center at $(2,180^\circ)$, vertices at $(1,180^\circ)$ and $(-3,0^\circ)$, the foci are at $(4,180^\circ)$ and at the pole, and the directrices are $r\cos\theta = -\frac{3}{2}$ and $r\cos\theta = -\frac{5}{2}$.



(j) The graph is the point $(2,-3)$.

(k) Hyperbola with center at $(0,0)$, vertices at $(\sqrt{2},0)$ and $(-\sqrt{2},0)$, foci at $(\sqrt{5},0)$ and $(-\sqrt{5},0)$, eccentricity of $\sqrt{\frac{5}{2}}$, directrices $x = \frac{2}{\sqrt{5}}$ and $x = -\frac{2}{\sqrt{5}}$, and asymptotes $y = \sqrt{\frac{3}{2}}x$ and $y = -\sqrt{\frac{3}{2}}x$.

(ℓ) Parabola with vertex at $(-2,3)$, focus at $(-4,3)$ and directrix $x = 0$.

(m) Ellipse with center at $(-2,-4)$, vertices at $(-8,-4)$ and $(4,-4)$, foci at $(-2 \pm \sqrt{11}, -4)$, eccentricity of $\frac{\sqrt{11}}{6}$, and directrices $x = -2 \pm \frac{36}{\sqrt{11}}$.

(n) Ellipse with center at $(1,-2)$, vertices at $(1 \pm \sqrt{5}, -2)$, eccentricity of $\frac{\sqrt{10}}{5}$, foci at $(1 \pm \sqrt{2}, -2)$, and directrices $x = 1 + \frac{5}{\sqrt{2}}$ and $x = 1 - \frac{5}{\sqrt{2}}$.

(o) The graph is the point $(3,-5)$.

(p) Hyperbola with center at $(-4,-1)$, vertices at $(-4 \pm \sqrt{27}, -1)$, foci at $(2,-1)$ and $(-10,-1)$, eccentricity of $\frac{2}{\sqrt{3}}$, directrices $x = \frac{1}{2}$ and $x = -\frac{17}{2}$, and asymptotes $x + 4 \pm \sqrt{3}(y + 1) = 0$.

(q) Hyperbola with center at $(-2,3)$, vertices at $(3,3)$ and $(-7,3)$, eccentricity of $\frac{13}{5}$, foci at $(11,3)$ and $(-15,3)$, directrices $x = -\frac{1}{13}$ and $x = \frac{-51}{13}$, and asymptotes $y = \frac{12}{5}x + \frac{39}{5}$ and $y = -\frac{12}{5}x - \frac{9}{5}$.

2. (a) $y^2 = -20x$

(b) $(x - 7)^2 = 32(y - 6)$

(c) Four circles, centers $(\pm 5, \pm 5)$.

Equations: $x^2 + y^2 - 10x \pm 10y + 25 = 0$, $x^2 + y^2 + 10x \pm 10y + 25 = 0$.

(d) $r = 2\sqrt{2}$. Equation: $x^2 + y^2 - 2x + 8y + 9 = 0$

(e) $C = (0,4)$, $r = 2\sqrt{5}$. Equation: $x^2 + y^2 - 8y - 4 = 0$

(f) $x^2 + y^2 - 12x + 8y - 48 = 0$

(g) $\frac{(x - 2)^2}{9} + \frac{4(y - 3)^2}{27} = 1$

(h) $\frac{(x + 3)^2}{2} + \frac{(y - 3)^2}{6} = 1$

(i) $\frac{(x - 2)^2}{4} + \frac{(y - 1)^2}{5} = 1$

(j) $\frac{-9x^2}{319} + \frac{16y^2}{319} = 1$

(k) $2x^2 - x - y + 5 = 0$

3. $(x + 1)^2 = 16y$

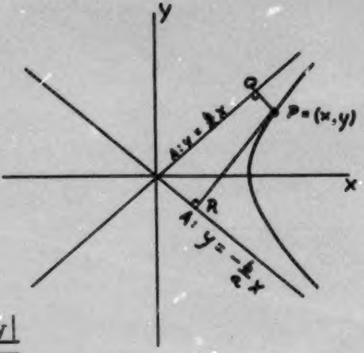
4. $(y + 1)^2 = -4(x - 2)$. Each center is equally distant from a fixed point $(1,-1)$ and a fixed line (the line $x = 3$).

5. $\frac{\sqrt{5}}{2}$

6. Elimination of the parameter gives $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

7. $3x^2 + 3y^2 + 6x - 14y + 6 = 0$.

8. Choose axes so that equation of curve is $b^2x^2 - a^2y^2 = a^2b^2$; then asymptotes are $y = \pm \frac{b}{a}x$. Then write expressions for distances from $P = (x,y)$ to asymptotes:



$$d(P,Q) \cdot d(P,R) = \frac{|bx - ay|}{\sqrt{a^2 + b^2}} \cdot \frac{|bx + ay|}{\sqrt{a^2 + b^2}}$$

$$= \frac{|b^2x^2 - a^2y^2|}{a^2 + b^2} = \frac{a^2b^2}{a^2 + b^2}$$

9. (a) If $\frac{b}{a} = 2$, $b = 2a$. Then $e = \frac{\sqrt{a^2 + b^2}}{a} = \frac{\sqrt{5a^2}}{a} = \sqrt{5}$

(b) In similar fashion, $e = \sqrt{1 + k^2}$

10. (a) Since $x^2 + y^2 = \frac{t^2r^2 + r^2}{1 + t^2}$, we get $x^2 + y^2 = r^2$.

(b) If only positive (or only negative) signs are used, the graph is only one-fourth of a circle; which part depends on the signs used, and also on whether r and t are positive or negative. If + signs are used, and r and t are both positive, it is the part in the first quadrant; if + signs are used, and $r > 0$, $t < 0$, it is the part in the second quadrant; and so on.

(c) In order for x to be $\pm r$, $\frac{t}{\sqrt{1 + t^2}} = \pm 1$. This is impossible, for $\sqrt{1 + t^2} = t$ only if $t = 0$, and then $x = 0$. Thus to be precise we would say that the parametric equations represent a circle with two points missing.

11. $e = \frac{\sqrt{a^2 + b^2}}{a}$, $e' = \frac{\sqrt{a^2 + b^2}}{b}$. $\frac{1}{e^2} + \frac{1}{e'^2} = \frac{a^2 + b^2}{a^2 + b^2} = 1$

12. $\frac{(x - a)^2}{k^2} + \frac{(y - b)^2}{k^2} = 1$. This is a circle with center (a,b) and radius $|k|$.

If k were zero, then the locus would be reduced to the point (a,b) .

13. Computed height at edge of road is $\frac{20\sqrt{5}}{3}$ ft. (≈ 14.9 ft.).

14. 20 feet

15. Let the equation of the hyperbola be $x^2 - y^2 = a^2$; then $e = \sqrt{2}$, $F = (ae,0)$, $F' = (-ae,0)$, and for a point (x,y) on the curve,

$$r = \sqrt{(x - ae)^2 + y^2} = \sqrt{(x - a\sqrt{2})^2 + x^2 - a^2} = \sqrt{(2x^2 + a^2) - 2a\sqrt{2}\,x}$$

similarly

$$r' = \sqrt{(2x^2 + a^2) + 2a\sqrt{2}\,x}$$

$$r \cdot r' = \sqrt{(2x^2 + a^2)^2 - 8a^2x^2}$$

$$= \sqrt{4x^4 - 4ax + a^4}$$

$$= |2x^2 - a^2| = 2x^2 - a^2 .$$

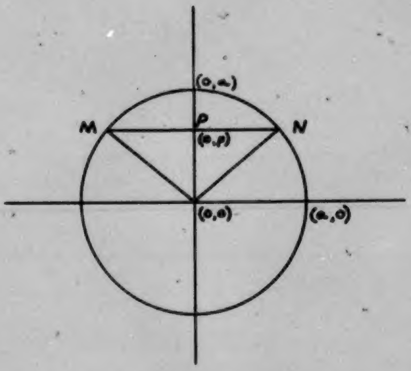
The square of the distance from the point to the center $(0,0)$ is $x^2 + y^2 = x^2 + x^2 - a^2 = 2x^2 - a^2$

16. (a) One possible form is $\frac{x^2}{a^2} + \frac{y^2}{\frac{16}{25}a^2} = 1$

(b) $\frac{4x^2}{225} + \frac{y^2}{36} = 1$

(c) $\frac{x^2}{25} + \frac{y^2}{16} = 1$

17. (a) The equation of the circle is $x^2 + y^2 = a^2$ and the equation of the chord is $y = p$. If $y = p$, then $x^2 = a^2 - p^2$ or $x = \pm\sqrt{a^2 - p^2}$



Then $d(P,N) = \sqrt{a^2 - p^2}$ and $d(P,M) = \sqrt{a^2 - p^2}$.

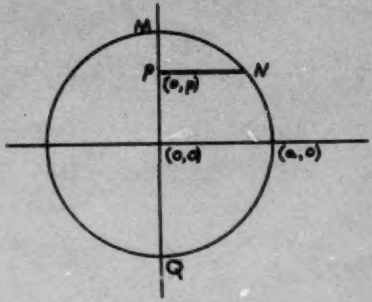
114

(b) $x^2 + y^2 = a^2$

$d(P,N) = \sqrt{a^2 - p^2}$, $d(M,P) = a - p$,

and $d(P,Q) = a + p$.

Then $\frac{d(M,P)}{d(P,N)} = \frac{d(P,N)}{d(P,Q)}$.



(c) Let (0,0) be one point and $P = (p,q)$ be the other.

$$\sqrt{(x - p)^2 + (y - q)^2} = k^2\sqrt{x^2 + y^2}$$

$$(k^2 - 1)x^2 + (k^2 - 1)y^2 + 2px + 2qy = p^2 + q^2$$

$$(x + \frac{p}{k^2 - 1})^2 + (y + \frac{q}{k^2 - 1})^2 = \frac{k^2(p^2 + q^2)}{(k^2 - 1)^2}$$

This is the equation of a circle.

We must restrict k so that is positive and not equal to one.

Challenge Problems

1. Let the hyperbola have equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Then the equations of the three lines named are, in order, $y = \frac{b}{a}x$, $x = \frac{a}{e}$, $y = \frac{-a}{b}(x - ae)$. These lines can be proved concurrent in any of a variety of ways.

2. If $P = (x,y)$ is the point where the explosion takes place, the 5-second time difference at A and B gives the condition

$$\sqrt{x^2 + y^2} - \sqrt{(x - 2)^2 + y^2} = 5(.2) ,$$

which becomes $$\frac{(x - 1)^2}{.25} - \frac{y^2}{.75} = 1 .$$

The 8-second difference at A and C gives the condition

$$\sqrt{x^2 + y^2} - \sqrt{x^2 + (y - 4)^2} = 8(.2) ,$$

which becomes $$\frac{(y - 2)^2}{.64} - \frac{x^2}{3.36} = 1 .$$

If we write equations of the appropriate asymptotes (the ones we want have positive slope), we have $y = \sqrt{3}(x - 1)$ and

$y = \frac{2}{\sqrt{21}} x + 2$. Solving these equations simultaneously, we find that the point of intersection is approximately $(2.9, 3.3)$. While a point of intersection of the asymptotes is not a point of intersection of the curves, it is probably satisfactory here since there was only one significant figure in the times given.

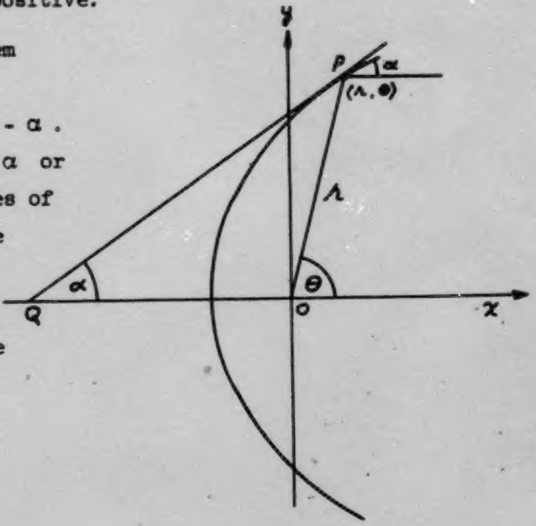
3. If the suggestion is followed, the condition is

$$d(P, W_2) - d(P, W_1) \geq 20$$

which is $\sqrt{(x + 15)^2 + y^2} - \sqrt{(x - 15)^2 + y^2} \geq 20$.

This becomes $\frac{x^2}{100} - \frac{y^2}{125} \geq 1$. The locus has as its boundary the part of the hyperbola for which x is positive.

4. From the statement of the problem and the diagram, we must show $m \angle OPQ = \alpha$. But $m \angle OPQ = \theta - \alpha$. Therefore we must show $\alpha = \theta - \alpha$ or $\theta = 2\alpha$. Rectangular coordinates of P are $(r \cos \theta, r \sin \theta)$; the equation of the parabola in rectangular coordinates is $y = 12(x + 3)$. The point-slope form of the equation of a line through P with slope m is



$$y - r \sin \theta = m(x - r \cos \theta) ,$$

or

$$y = mx + r(\sin \theta - m \cos \theta) .$$

This line will in general intersect the parabola in two points, but if it is a tangent line there will be just one such point. The coordinates of the intersection points can be found by solving simultaneously the equations of the line and parabola.

Thus by substituting we get a single equation for the x-coordinate,

$$\left(mx + r(\sin \theta - m \cos \theta)\right)^2 = 12(\alpha + 3) ,$$

or

$$m^2x^2 + \left(2mr(\sin\theta - m\cos\theta) - 12\right)x + r^2(\sin\theta - m\cos\theta)^2 - 36 = 0 .$$

Tangency requires that the roots of this equation be equal; therefore, the discriminant of the equation must equal zero. Hence

$$\left(2mr(\sin\theta - m\cos\theta) - 12\right)^2 - 4m^2\left(r^2(\sin\theta - m\cos\theta)^2 - 36\right) = 0 .$$

This equation can be eventually simplified to

$$3m^3 - mr(\sin\theta - m\cos\theta) + 3 = 0 .$$

But for this parabola $r = \frac{6}{1 - \cos\theta}$; substituting this in the equation just above, we obtain, with some more simplification,

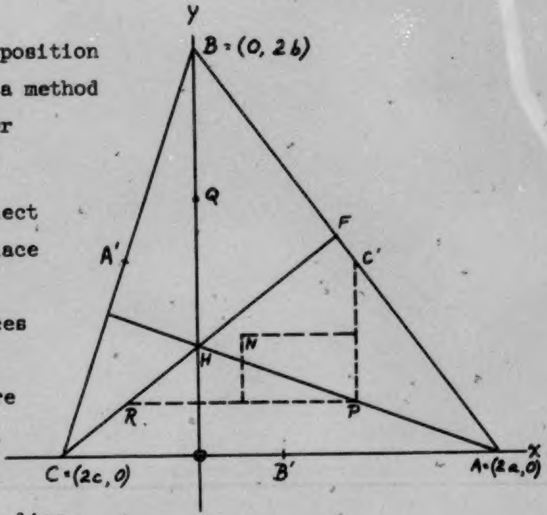
$$(1 + \cos\theta)m^2 - (2\sin\theta)m + 1 - \cos\theta = 0 .$$

Solving this for m gives the single value, $m = \frac{\sin\theta}{1 + \cos\theta}$.

But this is identically equal to $\tan\frac{\theta}{2}$. Since m is the tangent of the angle of inclination, $\angle$, we have $\alpha = \frac{\theta}{2}$ or $\theta = 2\alpha$, which is what we wanted to prove.

5. We indicate here one possible position of the triangle, and indicate a method of proof. There are many other possibilites.

In triangle ABC , we select one altitude as y-axis, and place the origin at the foot of the altitude. Then let the vertices be $A = (2a,0)$. $B = (0,2b)$, $C = (2c,0)$. The midpoints are $A' = (c,b)$, $B' = (a + c, 0)$, $C' = (a,b)$.



Altitude $\overline{CF}$ lies in the line with equation

$$y = \frac{a}{b}(x - 2c) .$$

It intersects altitude $\overline{BO}$ in point $H = (0, \frac{-2ac}{b})$. Hence the midpoints of $\overline{CH}$ and $\overline{AH}$ are $R = (c, \frac{-ac}{b})$ and $P = (a, -\frac{ac}{b})$.

The center of the circle through R, P, and C' would lie on the perpendicular bisectors of $\overline{RP}$ and $\overline{PC'}$; the point in which they intersect is

$$N = \left(\frac{a + c}{2}, \frac{b^2 - ac}{2b}\right).$$

Now we verify that the remaining six points (O, D, F, Q, B', A') lie on the same circle. One way would be to find the radius,

$$r = \frac{1}{2b}\sqrt{(a^2 + b^2)(b^2 + c^2)},$$

and verify that it is equal to the distance from N to each of these points.

Illustrative Test Items

1. Identify and sketch the curves whose equations are given.

(a) $r - 5 = 0$

(b) $r = 2 \sin \theta$

(c) $r = \frac{3}{1 - \cos \theta}$

(d) $r = \frac{4}{2 - \cos \theta}$

(e) $r = \frac{6}{4 - 8 \cos \theta}$

(f) $3y^2 - 4x^2 = 12$

(g) $x^2 - 4y - 4 = 0$

(h) $16x^2 + 25y^2 = 400$

(i) $9x^2 + 4y^2 - 36x + 32y + 100 = 0$

(j) $x^2 - 25y^2 + 2x + 100y - 99 = 0$

(k) $16x^2 - 9y^2 + 32x + 54y - 209 = 0$

(ℓ) $9x^2 + 4y^2 - 18x + 16y - 11 = 0$

2. Sketch the graphs of the following polar equations. Write the equations in rectangular form.

(a) $2r - 7 = 0$

(b) $r = \frac{3}{1 - 2 \cos \theta}$

(c) $r - r \sin \theta - 2 = 0$

(d) $r = \frac{5}{3 - 2 \cos \theta}$

3. Identify the following conic sections; give the eccentricity.

(a) $r = \frac{6}{3 - \cos \theta}$

(b) $2r - 3r \cos \theta - 12 = 0$

(c) $2r - 5 = 0$

(d) $r = 4 - r \cos \theta$

4. The directrix of a parabola is the line $y = x$, and the focus is $(4,-4)$. What are the coordinates of the vertex?

5. The eccentricity of a hyperbola is 2 and the distance between the foci is 8. Find the lengths of the semi-axes.

6. Write an equation of the tangent to the circle $x^2 + y^2 = 25$ at the point $(-3,4)$.

7. Find an equation of the radical axis of the circles with equations $(x - 3)^2 + (y + 2)^2 = 4$ and $x^2 + y^2 = 9$.

8. What kind of symmetry do the graphs of the following equations have? If there is point-symmetry, give the coordinates of the point; if line-symmetry, give an equation of each axis of symmetry.

(a) $r = \dfrac{6}{3 - 3\cos\theta}$

(b) $r = \cos\theta + \sin\theta$

(c) $25x^2 - 4y^2 = 100$

(d) $\dfrac{(x - 2)^2}{4} + \dfrac{(y + 4)^2}{36} = 1$

(e) $\dfrac{(x + 3)^2}{25} - \dfrac{(y - 1)^2}{4} = 1$

(f) $x^2 - 6x - y + 7 = 0$

9. Write an equation of a circle with center $(3,-1)$ and tangent to the line with the equation $2x + 5y - 5 = 0$.

10. The axes of an ellipse have lengths 10 and 6; what is its eccentricity?

11. Find the distance between the foci of the conic section with equation $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$.

12. The vertex of a parabola is $(1,5)$ and the focus is $(4,5)$. What is an equation of the directrix?

13. The directrix of a parabola is the line with equation $x = 2$, and the endpoints of the latus rectum are $(6,6)$ and $(6,-2)$. Write an equation of this parabola.

14. Write an equation of the circle having the segment with endpoints $(-1,3)$ and $(3,-3)$ as a diameter.

15. What is an equation of the conic with eccentricity of $\frac{3}{2}$ and foci at $(3,8)$ and $(3,2)$?

16. (a) Write an equation of the family of hyperbolas with center at $(2,-3)$ and asymptotes with slopes $\frac{2}{5}$ and $-\frac{2}{5}$.

(b) Find an equation of the member of that family which contains the point $(22,7)$.

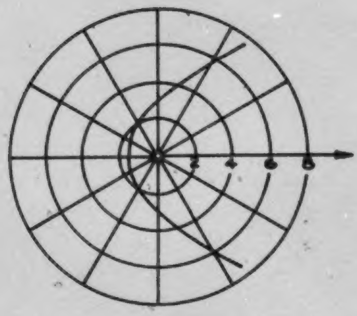
Answers for Illustrative Test Items

1. (In some routine cases the graphs are omitted.)

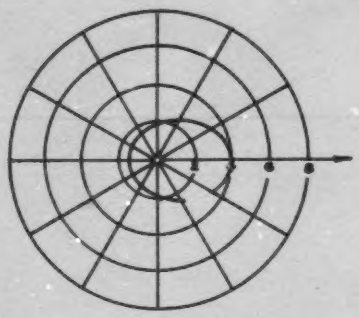
(a) Circle, center $(0,0)$, radius 5.

(b) Circle, center $(1,\frac{\pi}{2})$, radius 1.

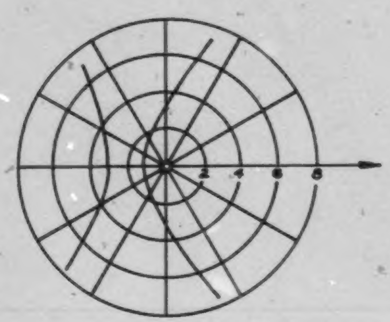
(c) Parabola $r = \frac{3}{1 - \cos \theta}$



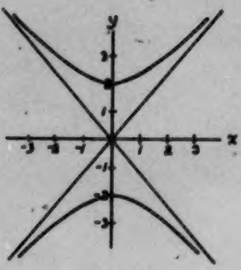
(d) Ellipse $r = \frac{2}{1 - .5 \cos \theta}$



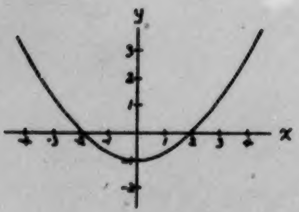
(e) Hyperbola $r = \frac{1.5}{1 - 2 \cos \theta}$



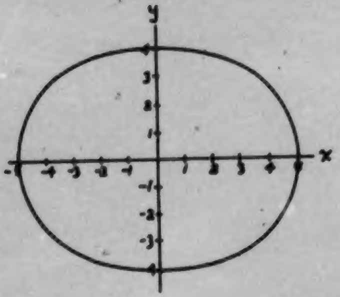
(f) **Hyperbola** $-\frac{x^2}{3} + \frac{y^2}{4} = 1$



(g) **Parabola** $x^2 = 4(y + 1)$



(h) Ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$



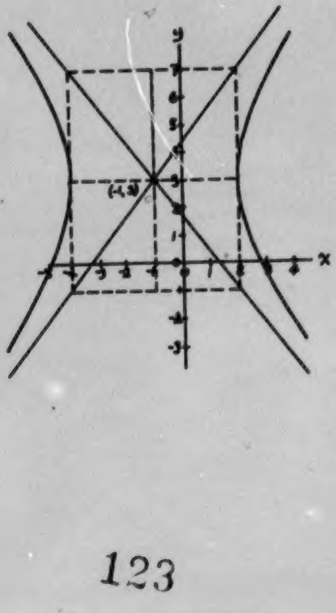
(i) Point-ellipse (2,-4)

$$9(x - 2)^2 + 4(y + 4)^2 = 0$$

(j) $(x + 1)^2 - 25(y - 2)^2 = 0$ or $(x + 5y - 9)(x - 5y + 11) = 0$

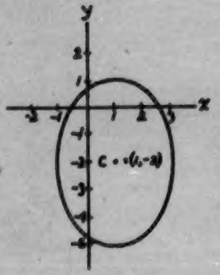
Two lines, equations $x + 5y - 9 = 0$ and $x - 5y + 11 = 0$.

(k) Hyperbola $\frac{(x + 1)^2}{9} - \frac{(y - 3)^2}{16} = 1$



(ℓ) $\frac{(x-1)^2}{4} + \frac{(y+2)^2}{9} = 1$

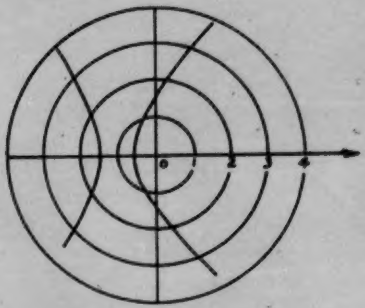
Ellipse



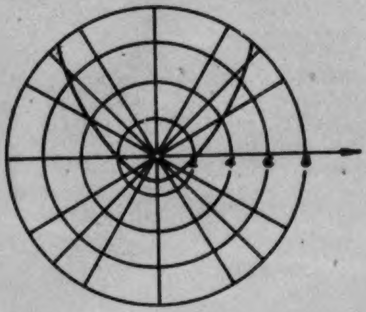
2. (a) Circle; $x^2 + y^2 = \frac{49}{4}$

(b) Hyperbola;

$$\frac{(x+2)^2}{1} - \frac{y^2}{3} = 1$$

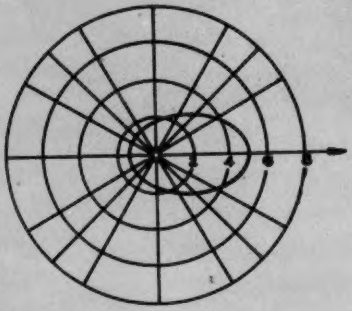


(c) **Parabola** $x^2 = 4(y + 1)$



(d) **Ellipse**

$$\frac{(x - 2)^2}{9} + \frac{y^2}{5} = 1$$



3. (a) **Ellipse;** $e = \frac{1}{3}$.

(b) **Hyperbola;** $e = \frac{3}{2}$.

(c) Circle; no eccentricity (or $e = 0$) .

(d) **Parabola;** $e = 1$

4. $V = (2,-2)$

5. $a = 2$, $b = 2\sqrt{3}$

6. $y - 4 = \frac{3}{4}(x + 3)$, or $3x - 4y + 25 = 0$

7. $3x - 2y - 9 = 0$

8. (a) Parabola; line symmetry, $\theta = 0$

   (b) Circle; point symmetry, $(\frac{\sqrt{2}}{2}, \frac{\pi}{4})$; line symmetry, every line through the center.

   (c) Hyperbola; point symmetry, $(0,0)$; line symmetry, $x = 0$ and $y = 0$.

   (d) Ellipse; point symmetry, $(2,-4)$; line symmetry, $x = 2$, $y = -4$.

   (e) Hyperbola; point symmetry, $(-3,1)$; line symmetry, $x = -3$ and $y = 1$.

   (f) Parabola; line symmetry, $x = 3$.

9. $(x - 3)^2 + (y + 1)^2 = \left(\frac{|2(3) + 5(-1) - 5|}{\sqrt{4 + 25}}\right)^2 = \frac{16}{29}$

   or $29x^2 + 29y^2 - 174x + 58y + 274 = 0$

10. $e = \frac{4}{5}$

11. $2ae = 2\sqrt{13}$

12. $x = -2$

13. $(y - 2)^2 = 8(x - 4)$

14. $C = (1,0)$; $r = \sqrt{13}$

    $(x - 1)^2 + y^2 = 13$

15. Hyperbola; $ae = 3$, $a = 2$, $b = \sqrt{5}$; $c = (3,5)$

$$-\frac{(x - 3)^2}{5} + \frac{(y - 5)^2}{4} = 1$$

16. (a) $\frac{(x - 2)^2}{25} - \frac{(y + 3)^2}{4} = k$

    (b) $\frac{(x - 2)^2}{25} - \frac{(y + 3)^2}{4} = -9$

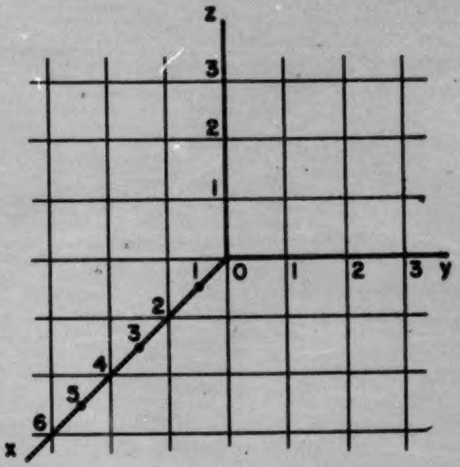
Teachers' Commentary

Chapter 8

THE LINE AND THE PLANE IN 3-SPACE

Parts of this chapter will be familiar to some classes. Time saved when this is the case may permit study of some of the supplementary chapters.

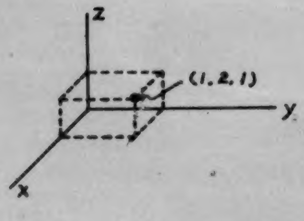
Many teachers have a favorite method of teaching students to make sketches of solids. If you do not have such a preferred method, you might like to try this. Have the students use squared paper; tell them to draw $\overleftrightarrow{OY}$ horizontal, $\overleftrightarrow{OZ}$ vertical, and $\overleftrightarrow{OX}$ at an angle of 45° with the negative end of the y-axis. Choose a suitable length for the unit on the y- and z-axis, and on the x-axis let the diagonal of the unit square measure two units. This is a convenient way of getting units, and makes a rather satisfactory drawing.



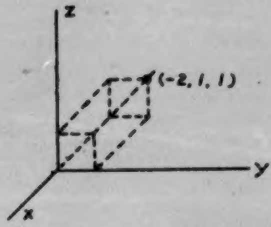
The formulas for point of division in Section 8-2 apply for both internal and external points of division. In Exercise 9, parts (b) to (f), of Exercises 8-2, the distances are considered to be directed, and two points are found.

Exercises 8-2

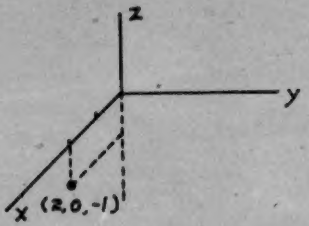
1. (a)



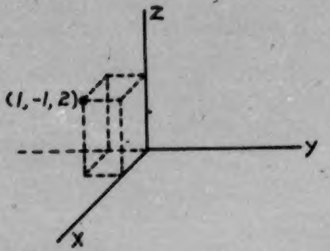
(b)



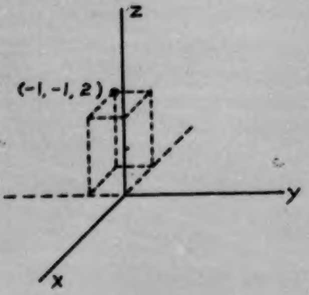
(c)



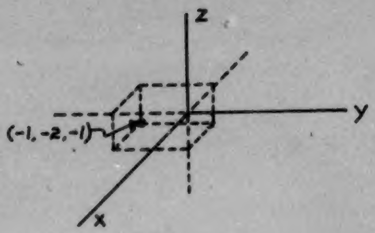
(d)



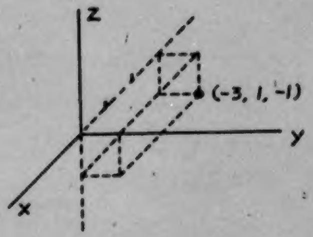
(e)



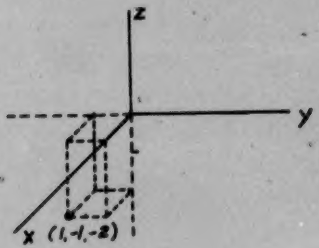
(f)



(g)



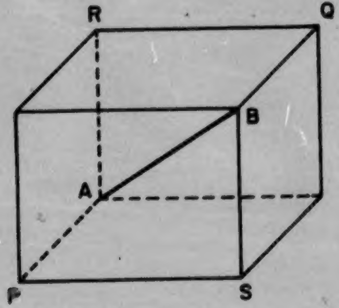
(h)



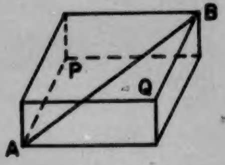
2. $d(O,P) = \sqrt{14}$, $d(O,Q) = \sqrt{14}$, $d(P,R) = \sqrt{30}$, $d(Q,R) = 5\sqrt{2}$.

3. Midpoint of $\overline{OP} = (\frac{1}{2}, 1, \frac{3}{2})$, midpoint of $\overline{PR} = (\frac{3}{2}, -\frac{1}{2}, 2)$.

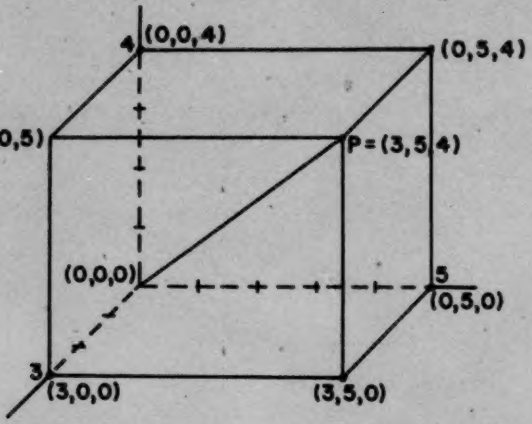
4. (a)



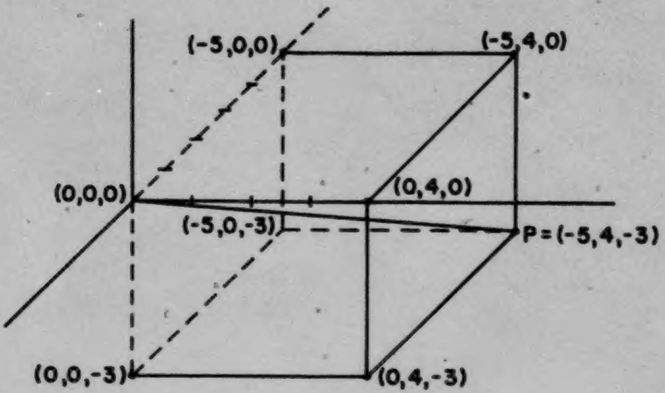
(b)



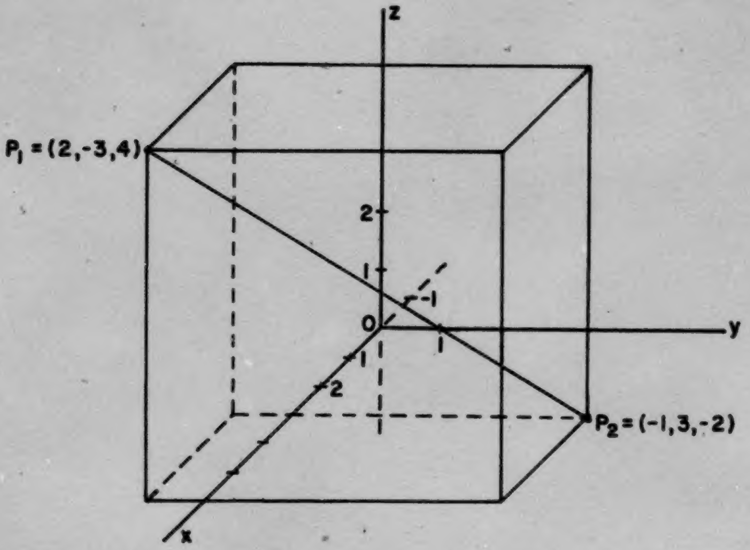
5.


4
(0,0,4)
(0,5,4)
(3,0,5)
P=(3,5,4)
(0,0,0)
5
(0,5,0)
3
(3,0,0)
(3,5,0)

6.


(-5,0,0)
(-5,4,0)
(0,0,0)
(0,4,0)
(-5,0,-3)
P=(-5,4,-3)
(0,0,-3)
(0,4,-3)

7. (a)


z
P₁=(2,-3,4)
2
1
-1
0
1
y
1
2
P₂=(-1,3,-2)
x

(b)

Projections	P_1	P_2
on x-axis	(2,0,0)	(-1,0,0)
on y-axis	(0,-3,0)	(0,3,0)
on z-axis	(0,0,4)	(0,0,-2)
on xy-plane	(2,-3,0)	(-1,3,0)
on yz-plane	(0,-3,4)	(0,3,-2)
on xz-plane	(2,0,4)	(-1,0,-2)

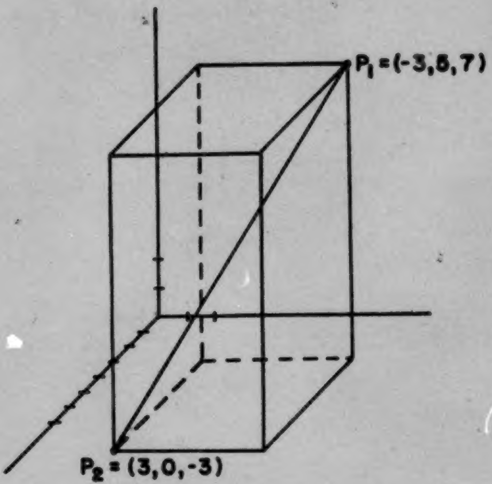
(c) $d(P_1,P_2) = 9$

lengths of projections

on x-,y-, and z-axes: 3,6,6

on xy-, yz-, and xz-planes: $3\sqrt{5}$, $6\sqrt{2}$, $3\sqrt{5}$

8. (a)



(b)

Projections	P_1	P_2
on x-axis	(-3,0,0)	(3,0,0)
on y-axis	(0,5,0)	(0,0,0)
on z-axis	(0,0,7)	(0,0,-3)
on xy-plane	(-3,5,0)	(3,0,0)
on yz-plane	(0,5,7)	(0,0,-3)
on xz-plane	(-3,0,7)	(3,0,-3)

(c) $d(P_1, P_2) = \sqrt{161}$

lengths of projections

on x- , y- , and z-axes 6, 5, 10

on xy- , yz- , and xz-planes $\sqrt{61}, 5\sqrt{5}, 2\sqrt{34}$

9. (a) $(\frac{1}{2}, -\frac{1}{2}, 2)$

(b) $(\frac{4}{3}, -\frac{5}{3}, \frac{10}{3})$ or $(8, -11, 14)$

(c) $(\frac{9}{8}, -\frac{11}{8}, 3)$ or $(\frac{21}{2}, -\frac{29}{2}, 18)$

(d) $(-\frac{1}{8}, \frac{3}{8}, 1)$ or $(-\frac{19}{2}, \frac{27}{2}, -14)$

(e) $(0, \frac{1}{5}, \frac{6}{5})$ or $(6, -\frac{41}{5}, \frac{54}{5})$

(f) $(\frac{34}{3}, -\frac{47}{3}, \frac{58}{3})$ or $(-\frac{16}{3}, \frac{23}{3}, -\frac{22}{3})$

10. $d(A,B) = \sqrt{14}$

$d(A,C) = \sqrt{34}$

$d(B,C) = \sqrt{20}$

Right triangle.

Challenge Problem

The question of how we know there are three mutually perpendicular lines through a point in space is intended as a warning to the students against the uncritical use of intuition. It is not a trivial question. In terms of the development in the SMSG Geometry it can be answered as follows. By a postulate, there are at least four points in space, so we can select O and another point, P . By another postulate there is a unique line L_1 , containing O and P . By a theorem, there exists a unique plane α through O perpendicular to $\overleftrightarrow{OP}$. By a postulate there is another point Q in α . By a postulate there is a unique line L_2 through O and Q , and it is perpendicular to $\overleftrightarrow{OP}$. Finally, by a theorem there is a unique line in α through O perpendicular to L_2 , and by another theorem it is perpendicular to L_1 too.

The argument by which the parametric representation is obtained is rather tricky and should probably be gone over in class very carefully. It may help to show that

$$z = z_0 + s(z_1 - z_0)$$

for suitable s by noting that $z_1 - z_0 \neq 0$ and hence $s = \frac{z - z_0}{z_1 - z_0}$ will do. In the final step the argument is that from the parametric equations for L' and L" we see that

$$y = y_0 + s(y_1 - y_0)$$

for suitable t and that

$$y = y_0 + s(y_1 - y_0)$$

for suitable s. Since $y_1 - y_0 \neq 0$,

$$s = \frac{y - y_0}{y_1 - y_0} = t.$$

Students are often intrigued by the idea of a 4-dimensional space, so they may enjoy our brief discussion of the notion. If it is taken up, you should try to make it clear that we are not introducing a coordinate system into a space which is given (by a system of postulates) but instead are defining a "space" which is in many ways like the space of ordinary geometry.

In 3-space, as in 2-space, a line has not just one, but many representations. Only one is given in this commentary, except where the directions specifically ask for two. A student should be allowed to write any correct representation, but should be able to show that his representation is equivalent to any desired representation.

In Exercise 2 of Exercises 8-3, you may want to have the students consider further the cases in which symmetric representation is not possible. In part (d), for example, since one of the direction numbers is zero, symmetric equations cannot be written. However, t can be eliminated and the equations of two planes containing the line written: $z - 1 = 0$, $x + 5y + 13 = 0$. In part (a), with two direction numbers equal to zero, we have at once the equations of two such planes: $y = 2$, $z = 3$.

Exercises 8-3

1. (a) $$\begin{cases} x = 1 + t(1) \\ y = 2 + t(0) \\ z = 3 + t(0) \end{cases}$$

(b) $$\begin{cases} x = -3 + t(0) \\ y = -2 + t(0) \\ z = 1 + t(1) \end{cases}$$

(c) $$\begin{cases} x = 1 + t(-4) \\ y = 2 + t(-4) \\ z = 3 + t(-2) \end{cases}$$

(d) $$\begin{cases} x = -3 + t(5) \\ y = -2 + t(-1) \\ z = 1 + t(0) \end{cases}$$

(e) $$\begin{cases} x = 0 + t(-4) \\ y = 0 + t(-4) \\ z = 0 + t(-2) \end{cases}$$

(f) $$\begin{cases} x = 0 + t(5) \\ y = 0 + t(-1) \\ z = 0 + t(0) \end{cases}$$

(g) $$\begin{cases} x = 0 + t(1) \\ y = 0 + t(2) \\ z = 0 + t(3) \end{cases}$$

(h) $$\begin{cases} x = 1 + t(-1) \\ y = 2 + t(-2) \\ z = 3 + t(0) \end{cases}$$

(i) $$\begin{cases} x = 1 + t(5) \\ y = 2 + t(-1) \\ z = 3 + t(0) \end{cases}$$

(j) $$\begin{cases} x = 2 + t(-4) \\ y = -3 + t(-4) \\ z = 1 + t(-2) \end{cases}$$

2. The symmetric form exists in parts (c), (e), (g) and (j)

(c) $\frac{x - 1}{-4} = \frac{y - 2}{-4} = \frac{z - 3}{-2}$

(e) $\frac{x}{-4} = \frac{y}{-4} = \frac{z}{-2}$

(g) $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$

(j) $\frac{x - 2}{-4} = \frac{y + 3}{-4} = \frac{z - 1}{-2}$

3. (a) $(1,0,0)$

(b) $(0,0,1)$

(c) $(-\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3})$

(d) $(\frac{5}{\sqrt{26}}, -\frac{1}{\sqrt{26}}, 0)$

(e) $(-\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3})$

(f) $(\frac{5}{\sqrt{26}}, -\frac{1}{\sqrt{26}}, 0)$

(g) $(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}})$

(h) $\left(-\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}, 0\right)$

(i) $\left(\frac{5}{\sqrt{26}}, -\frac{1}{\sqrt{26}}, 0\right)$

(j) $\left(-\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}\right)$

4. (a) $$\begin{cases} x = 1 + t(-1) \\ y = 1 + t(-2) \\ z = -2 + t(1) \end{cases} \qquad \begin{cases} x = 0 + t(1) \\ y = -1 + t(2) \\ z = -1 + t(-1) \end{cases} \qquad (-1, -3, 0)$$

(b) $$\begin{cases} x = -1 + t(-1) \\ y = -1 + t(0) \\ z = -1 + t(2) \end{cases} \qquad \begin{cases} x = -2 + t(1) \\ y = -1 + t(0) \\ z = 1 + t(-2) \end{cases} \qquad (-3, -1, 3)$$

(c) $$\begin{cases} x = 4 + t(-3) \\ y = 2 + t(-4) \\ z = 1 + t(3) \end{cases} \qquad \begin{cases} x = 1 + t(3) \\ y = -2 + t(4) \\ z = 4 + t(-3) \end{cases} \qquad (-2, -6, 7)$$

(d) $$\begin{cases} x = -3 + t(4) \\ y = 1 + t(1) \\ z = 1 + t(-2) \end{cases} \qquad \begin{cases} x = 1 + t(-4) \\ y = 2 + t(-1) \\ z = -1 + t(2) \end{cases} \qquad (5, 3, -3)$$

5. (a) $\left(\frac{-1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$; 114°, 145°, 66°. $\left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right)$; 66°, 35°, 114°.

(b) $\left(\frac{-1}{\sqrt{5}}, \frac{0}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$; 117°, 90°, 27°. $\left(\frac{1}{\sqrt{5}}, \frac{0}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right)$; 63°, 90°, 153°.

(c) $\left(\frac{-3}{\sqrt{34}}, \frac{-4}{\sqrt{34}}, \frac{3}{\sqrt{34}}\right)$; 121°, 133°, 59°. $\left(\frac{3}{\sqrt{34}}, \frac{4}{\sqrt{34}}, \frac{-3}{\sqrt{34}}\right)$; 59°, 47°, 121°.

(d) $\left(\frac{4}{\sqrt{21}}, \frac{1}{\sqrt{21}}, \frac{-2}{\sqrt{21}}\right)$; 29°, 77°, 116°. $\left(\frac{-4}{\sqrt{21}}, \frac{-1}{\sqrt{21}}, \frac{2}{\sqrt{21}}\right)$; 151°, 103°, 64°.

6. x-axis (1,0,0)
y-axis (0,1,0)
z-axis (0,0,1)

7. $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

8. (a) No
   (b) Yes
   (c) Yes
   (d) Yes

9. Lines with equations (a) and (f) are parallel; so are (b) and (d) .

10. $L_1 : \frac{x - 2}{3} = \frac{y - 1}{-2} = \frac{z + 1}{-1}$ $\qquad$ $L_2 : \frac{x + 1}{1} = \frac{y - 2}{2} = \frac{z - 4}{-1}$

$L_3 : \frac{x - 3}{2} = \frac{y + 5}{-3} = \frac{z}{4}$

L_4 cannot be written in symmetric form.

11. $$d(P_1, P_2) = \sqrt{[(x_0 + \ell t_1) - (x_0 + \ell t_2)]^2 + [(y_0 + mt_1) - (y_0 + mt_2)]^2 + [(z_0 + nt_1) - (z_0 + nt_2)]^2}$$

$$= \sqrt{(\ell t_1 - \ell t_2)^2 + (mt_1 - mt_2)^2 + (nt_1 - nt_2)^2}$$

$$= \sqrt{(\ell^2 + m^2 + n^2)(t_1 - t_2)^2}$$

$$= \sqrt{\ell^2 + m^2 + n^2}\ |t_1 - t_2|$$

The distance between any two points on a line with the given parametirc representation is a constant multiple $(\sqrt{\ell^2 + m^2 + n^2})$ of the absolute value of the difference of the values of the parameter that give the points. If the direction numbers are normalized, the distance is equal to the absolute value of the difference of the parameters.

12. Suppose L is in or parallel to the xy-plane. In that plane, L would have the parametric representation

$$x = x_0 + t(x_1 - x_0)$$
$$y = y_0 + t(y_1 - y_0) .$$

The z-coordinate of every point on L would be the same number, so $z = z_0$. Thus Equations (3) would represent the line L ; similarly, they would represent a line parallel to either of the other coordinate planes.

Challenge Problems

1. For all values of t the x-coordinate of the point on L will be 2 so the line is in the plane $x = 2$. Similarly for every point $P(x,y,z)$ in the plane $3y - z = -5$ there is a value of t such that $y = -1 + t$ and $z = 2 + 3t$.

   So $L : x = 2$, $y = -1 + t$, $z = 2 + 3t$ lies in the intersection of the planes $x = 2$ and $3y - z = -5$.

2. $x = 2 \quad z = -1$

3. $$\begin{cases} x = x_0 + t(x_1 - x_0) \\ y = y_0 + t(y_1 - y_0) \\ z = z_0 + t(z_1 - z_0) \\ w = w_0 + t(w_1 - w_0) \end{cases}$$

   If $P_2(x_2, y_2, z_2, w_2)$ is on L then there is a number

   $$t_2 \neq 0 \text{ such that } \begin{cases} x_2 = x_0 + t_2(x_1 - x_0) \\ y_2 = y_0 + t_2(y_1 - y_0) \\ z_2 = z_0 + t_2(z_1 - z_0) \\ w_2 = w_0 + t_2(w_1 - w_0) \end{cases}$$

   But tnen

   $$\begin{cases} x_1 = x_0 + (\frac{1}{t_2})(x_2 - x_0) \\ y_1 = y_0 + (\frac{1}{t_2})(y_2 - y_0) \\ z_1 = z_0 + (\frac{1}{t_2})(z_2 - z_0) \\ w_1 = w_0 + (\frac{1}{t_2})(w_2 - w_0) \end{cases}$$

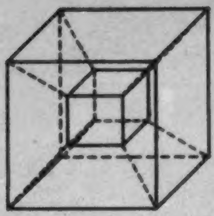
   So P_1 is on the line through P_0 and P_2.

4. On the coordinate axes, $(x_0,0,0,0)$, $(0,y_0,0,0)$, $(0,0,z_0,0)$, $(0,0,0,w_0)$.

   On the coordinate planes, $(x_0,y_0,0,0)$, $(0,y_0,z_0,0)$, $(0,0,z_0,w_0)$, $(x_0,0,0,w_0)$, $(x_0,0,z_0,0)$, $(0,y_0,0,w_0)$.

   On the coordinate hyperplanes, $(x_0,y_0,z_0,0)$, $(0,y_0,z_0,w_0)$, $(x_0,0,y_0,z_0)$, $(x_0,y_0,0,w_0)$.

5.



In 3-space $V - E + F = 2$ where V is the number of vertices, E, the number of edges, F, the number of faces.

In 4-space the polyhedron is made up of vertices (0-dimensional), edges (1-dimensional), faces (2-dimensional), and hyperfaces (3-dimensional). In the picture the hyperfaces are represented as truncated pyramids with bases on faces of the figures that appear as inner and outer cubes.
$V - E + F - H = 0$

Exercises 8-4

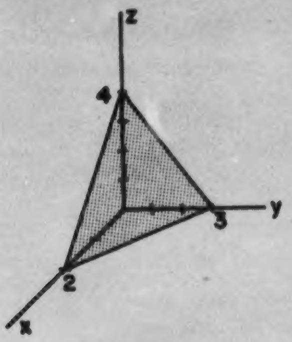
1. $2x - y - 4z + 6 = 0$

2. $x + 4y - 5z - 6 = 0$

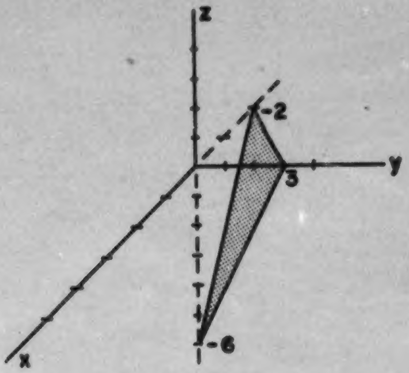
3.

	Intercepts			Traces in		
	x-	y-	z-	xy-plane	yz-plane	xz-plane
(a)	2	3	4	$3x + 2y - 6 = 0$	$4y + 3z - 12 = 0$	$2x + z - 4 = 0$
(b)	5	2	10	$2x + 5y - 10 = 0$	$5y + z - 10 = 0$	$2x + z - 10 = 0$
(c)	$2\frac{1}{2}$	-5	-2	$2x - y - 5 = 0$	$2y + 5z + 10 = 0$	$4x - 5z - 10 = 0$
(d)	-2	3	-6	$3x - 2y + 6 = 0$	$-2y + z + 6 = 0$	$3x + z + 6 = 0$
(e)	4	-3	none	$3x - 4y - 12 = 0$	$y + 3 = 0$	$x - 4 = 0$
(f)	none	-4	$\frac{5}{2}$	$y + 4 = 0$	$5y - 8z + 20 = 0$	$-2z + 5 = 0$
(g)	0	0	0	$x - 2y = 0$	$-3y + z = 0$	$3x + 2z = 0$
(h)	none	0	0	$y = 0$	$3y - 5z = 0$	$z = 0$
(i)	7	none	none	$x - 7 = 0$	none	$x - 7 = 0$
(j)	none	none	$-\frac{9}{2}$	none	$2z + 9 = 0$	$2z + 9 = 0$

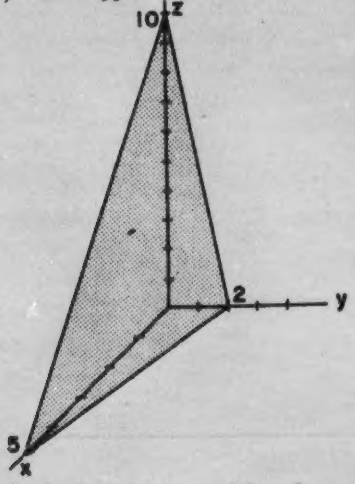
(a) $6x + 4y + 3z - 12 = 0$



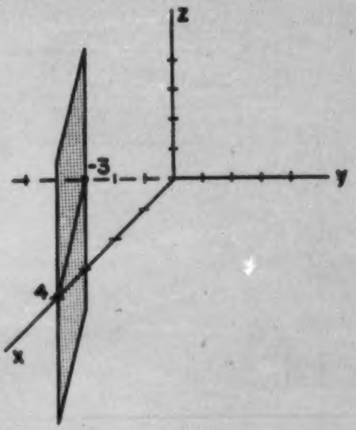
(d) $3x - 2y + z + 6 = 0$



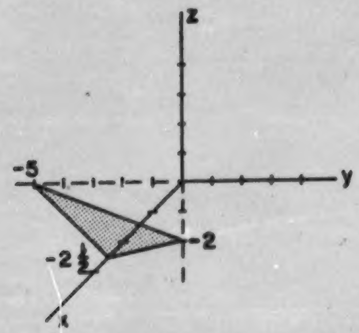
(b) $2x + 5y + z - 10 = 0$



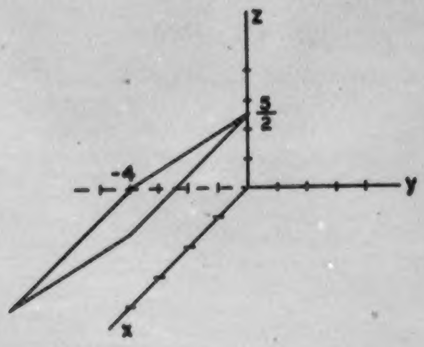
(e) $3x - 4y - 12 = 0$



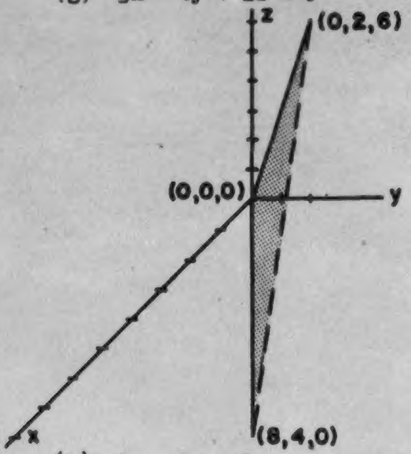
(c) $4x - 2y - 5z - 10 = 0$


z
-5
y
-2
$-2\frac{1}{2}$
x

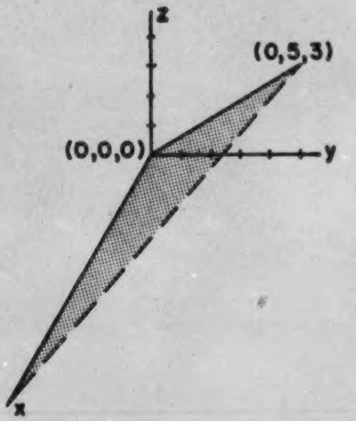
(f) $5y - 8z + 20 = 0$


z
$\frac{5}{2}$
-4
y
x

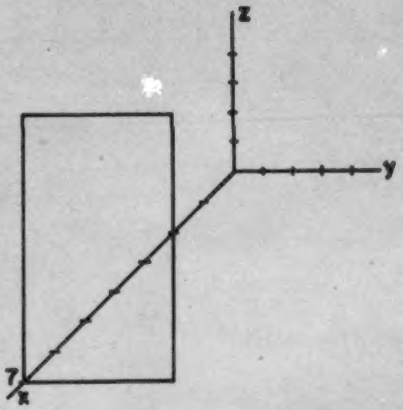
(g) $3x - 6y + 2z = 0$


z
(0,2,6)
(0,0,0)
y
x
(8,4,0)

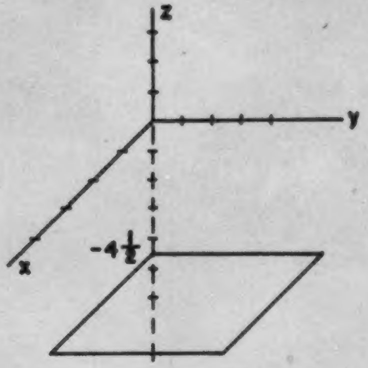
(h) $3y - 5z = 0$



(i) $x - 7 = 0$

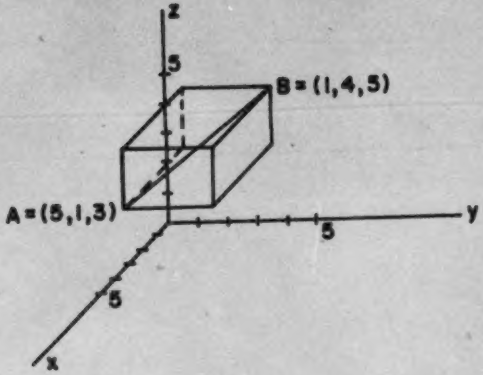


(j) $2z + 9 = 0$

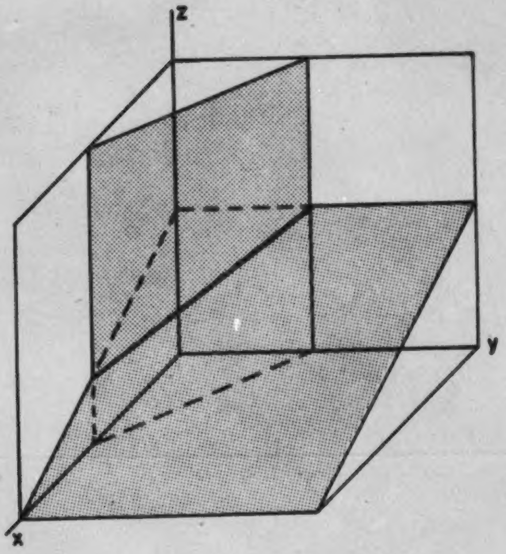

z
y
x
$-4\frac{1}{2}$

4. (a) $ax + by + cz = 0$
   (b) $cz + d = 0$
   (c) $ax + d = 0$
   (d) $ax + by + d = 0$
   (e) $by + cz + d = 0$
   (f) $ax + cz + d = 0$

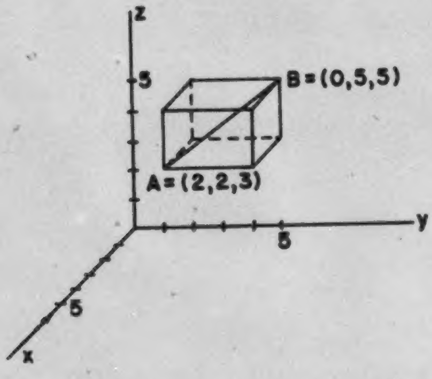
5. (a)


z
5
B = (1,4,5)
A = (5,1,3)
y
5
5
x

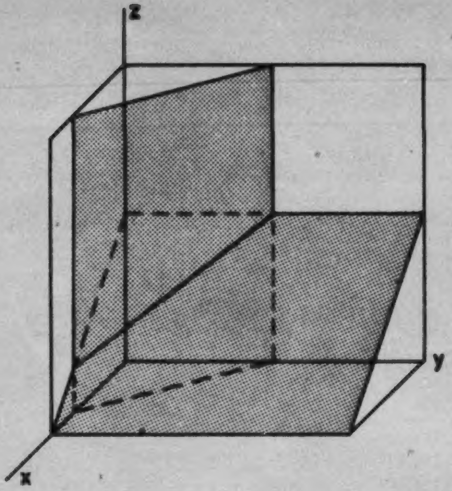
(b)



6. (a)


z
5
B = (0,5,5)
A = (2,2,3)
y
5
5
x

(b)



7. $(\ell,m,n) = (3,-2,5)$; $(\lambda, \mu, \nu) = (\frac{3}{\sqrt{38}}, \frac{-2}{\sqrt{38}}, \frac{5}{\sqrt{38}})$

8. $(\ell,m,n) = (4,-1,0)$; $(\lambda, \mu, \nu) = (\frac{4}{\sqrt{17}}, \frac{-1}{\sqrt{17}}, 0)$

9. (a) $\frac{4}{\sqrt{61}}$

(b) 0

(c) $\frac{28}{\sqrt{45}}$

(d) $\frac{1}{\sqrt{14}}$

(e) $\frac{23}{5}$

(f) $\frac{14}{\sqrt{89}}$

(g) $\frac{11}{7}$

(h) $\frac{4}{\sqrt{34}}$

(i) 8

(j) $\frac{13}{2}$

10. (a) $\frac{7}{\sqrt{61}}$

(b) $\frac{11}{\sqrt{30}}$

(c) $\frac{3}{\sqrt{5}}$

(d) 0

(e) 5

(f) $\frac{49}{\sqrt{89}}$

(g) $\frac{23}{7}$

(h) $\frac{17}{\sqrt{34}}$

(i) 6

(j) $\frac{7}{2}$

11. (a) $-8x + 3y - 7z + 23 = 0$
    (b) $2x - 3z - 1 = 0$

12. (a) $3x - 2y + z + 4 = 0$
    (b) $x - 2z + 5 = 0$

13. Let the equation of the plane be $Ax + By + Cz + D = 0$.
Since the plane contains $(a,0,0)$, $(0,b,0)$, and $(0,0,c)$, we have

$$Aa + D = 0 \qquad \text{or} \qquad A = -\frac{D}{a} ,$$

$$Bb + D = 0 \qquad \text{or} \qquad B = -\frac{D}{b} ,$$

and $$Cc + D = 0 \qquad \text{or} \qquad C = -\frac{D}{c} .$$

Thus if $D = -1$, an equation is $\frac{x}{a} + \frac{y}{b} + \frac{c}{z} = 1$.

14. (a) $\frac{x}{1} + \frac{y}{3} + \frac{z}{4} = 1$

    (b) $\frac{x}{-2} + \frac{y}{5} + \frac{z}{-3} = 1$

15. (a) $x - 12y + 3z - 7 = 0$
    (b) $3x + 8y + 2z - 15 = 0$

16. One method would be to find an equation of the plane determined by points A , B , C $(2x - y + z - 1 = 0)$ and then check that point D is a point of this plane.

17. (a) $x + y + 2z - 2 = 0$
    (b) $3y - 2z + 1 = 0$

18. The proof given in Intermediate Mathematics may be familiar; it follows.

Let $P = (x,y,z)$ be any point on the plane that is the set of points equidistant from $O = (0,0,0)$ and $Q = (ka,kb,kc)$ where

$$k = \frac{-2d}{a^2 + b^2 + c^2}$$

Then, since $d(P,O) = d(P,Q)$,

$$x^2 + y^2 + z^2 = (x - ka)^2 + (y - kb)^2 + (z - kc)^2 ,$$

or $$0 = -2kax + k^2a^2 - 2kby + k^2b^2 - 2kc + k^2c^2 ,$$

which becomes $$2k(ax + by + cz) = k^2(a^2 + b^2 + c^2)$$

or $$ax + by + cz = \frac{k}{2}(a^2 + b^2 + c^2) .$$

By substituting the value of k, this equation becomes

$$ax + by + cz + d = 0 .$$

This argument is reversible. This means that any point P whose coordinates satisfy $ax + by + cz + d = 0$ is equidistant from the points O and Q. Hence $ax + by + cz + d = 0$ is the equation of a plane.

Note. If $d = 0$, it follows that $k = 0$; the two points coincide, and no plane is determined. In this case we use the symmetric points (a,b,c) and $(-a,-b,-c)$ and carry through the same steps as above.

8-5 The definition of a vector as a set of equivalent directed segments makes the extension to 3-space almost trivial. Since any member of the set may represent the vector, we are free to choose those representatives which most simplify our models or diagrams. In Chapter 3 we stressed the freedom, but we did not attempt initially to pursue all the consequences of this freedom. At this point it may be helpful to review the earlier material briefly and to point out that all vectors which have representatives on parallel lines also have representatives on the same line. Once this property of vectors is understood, the approach to 3-space should follow more easily.

The proof of Example 1 assumes no more knowledge of prisms than the material presented in the SMSG Geometry. If students have had additional training in solid geometry, they should be able to develop a more concise proof.

In Chapter 3 we approached vectors from a purely geometric point of view before introducing any analysis using components. Here we have adopted the same approach, but in almost any application of vectors it is more convenient to use representations in component form.

For simplicity we use i, j, and k to represent basis vectors without the usual symbols indicating vector quantities. Consequently it may be necessary to stress that these are indeed vectors.

Exercises 8-5

1. (a) 0
   (b) 0
   (c) 0
   (d) 1
   (e) 1
   (f) 1
   (g) 0
   (h) 7

2. (a) 0
   (b) 0
   (c) 0
   (d) 1
   (e) 1
   (f) 1
   (g) 0
   (h) $\frac{\sqrt{7}}{2\sqrt{3}}$ or $\frac{\sqrt{21}}{6}$

3. $r = \pm \frac{1}{3}$

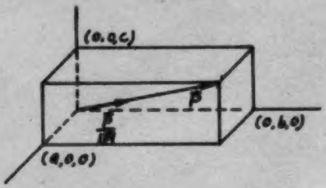
4. (a) [14,-3,3]
   (b) [-7,16,-9]
   (c) [-2,17,-9]
   (d) [6,0,2]
   (e) [14,10,-2]
   (f) [-18,-4,-6]

5. (a) [6,-2,2]
   (b) $[\frac{17}{5}, -\frac{12}{5}, \frac{9}{5}]$
   (c) $[-\frac{1}{3}, -\frac{1}{3}, -\frac{2}{3}]$
   (d) $[0, -\frac{2}{3}, 0]$
   (e) [-7,0,1]
   (f) $[-\frac{13}{6}, 0, -\frac{1}{6}]$

6. (a) -1
   (b) -6
   (c) 24
   (d) -50
   (e) 0
   (f) -70
   (g) 257
   (h) 4
   (i) -231
   (j) 42

7. $\vec{A}\cdot\vec{A}$ is a real number defined $|\vec{A}||\vec{A}| \cos\theta$ where θ is the angle between $\vec{A}$ and $\vec{A}$; i.e., $\theta = 0^\circ$. So $\vec{A}\cdot\vec{A} = |\vec{A}|\cdot|\vec{A}| = |\vec{A}|^2$. $|\vec{A}|^2$ and $|\vec{A}|^3$ are real numbers. $\vec{A}\cdot\vec{A}\cdot\vec{A}$ is not defined unless a convention about the order of mutliplication is made, but in any case, e.g., $(\vec{A}\cdot\vec{A})\cdot\vec{A} = |\vec{A}|^2\,\vec{A}$, the product is a vector, not a number.

8. $\frac{P}{|P|} = \frac{a}{\sqrt{a^2 + b^2 + c^2}}\, i + \frac{b}{\sqrt{a^2 + b^2 + c^2}}\, j + \frac{c}{\sqrt{a^2 + b^2 + c^2}}\, k$.



A vector of magnitude one in the direction of P .

9. $k = -\frac{9}{2}$

10. $\frac{1}{\sqrt{3}}$

11. The line segment joining the endpoint of $\vec{A}$ and $\vec{B}$ is parallel to $\vec{A} - \vec{B}$. But $\vec{A} - \vec{B} = i + 4j$ which lies in the xy-plane. Hence $\overleftrightarrow{AB}$ is parallel to the xy-plane.

12. If $\vec{c} \perp \vec{a}$ and $\vec{c} \perp \vec{b}$, we must show that $\vec{c} \perp (\vec{a} + \vec{b})$ or that $\vec{c} \cdot (\vec{a} + \vec{b}) = 0$. But $\vec{c} \cdot (\vec{a} + \vec{b}) = \vec{c} \cdot \vec{a} + \vec{c} \cdot \vec{b}$. Since $\vec{c} \perp \vec{a}$, $\vec{c} \cdot \vec{a} = 0$, and since $\vec{c} \perp \vec{b}$, $\vec{c} \cdot \vec{b} = 0$. Therefore, $\vec{c} \cdot (\vec{a} + \vec{b}) = 0$ and $\vec{c} \perp (\vec{a} + \vec{b})$.

13. $[0,0,\pm 1]$ or $\pm k$.

14. $-7i + 6j - k$
$7ci - 6cj + ck$, where $c \neq 0$.

15. $\angle A = 124^\circ$
$\angle B = 32^\circ$
$\angle C = 24^\circ$

16. $\vec{P} + r[0,1,-\frac{b}{c}]$ is a parametric representation of a line through $\vec{P} = [a,b,c] \neq [0,0,0]$ which is perpendicular to $\vec{P}$; i.e., there is such a line for each r .

17. We wish to prove that $r\vec{A} = [ra_1, ra_2, ra_3]$.

Since $\vec{A} = [a_1,a_2,a_3]$,
$r\vec{A} = r[a_1,a_2,a_3]$.

Does $r[a_1,a_2,a_3] = [ra_1,ra_2,ra_3]$?

If two vectors are equal, their magnitudes and directions are equal.

magnitude of $r[a_1,a_2,a_3] = |r|\sqrt{a_1^2 + a_2^2 + a_3^2}$

magnitude of $[ra_1,ra_2,ra_3] = \sqrt{r^2a_1^2 + r^2a_2^2 + r^2a_3^2}$

$$|r|\sqrt{a_1^2 + a_2^2 + a_3^2} .$$

If r is positive or negative the directions of the two are equal. Therefore the vectors are equal.

18. (a) We wish to prove that

$$\vec{X}\cdot(\vec{Y} + \vec{Z}) = \vec{X}\cdot\vec{Y} + \vec{X}\cdot\vec{Z} .$$

We expand the left-hand member to obtain

$$\begin{aligned}
&[x_1,x_2,x_3]\cdot[y_1 + z_1, y_2 + z_2, y_3 + z_3] \\
&= x_1(y_1 + z_1) + x_2(y_2 + z_2) + x_3(y_3 + z_3) \\
&= x_1y_1 + x_1z_1 + x_2y_2 + x_2z_2 + x_3y_3 + x_3z_3 \\
&= x_1y_1 + x_2y_2 + x_3y_3 + x_1z_1 + x_2z_2 + x_3z_3 \\
&= \vec{X}\cdot\vec{Y} + \vec{X}\cdot\vec{Z} .
\end{aligned}$$

This is the right-hand side of the equation and the proof is complete.

(b) To prove $(t\vec{X})\cdot\vec{Y} = t(\vec{X}\cdot\vec{Y})$.

The left-hand member is expanded to obtain

$$\begin{aligned}
&[tx_1,tx_2,tx_3]\cdot[y_1,y_2,y_3] \\
&= tx_1y_1 + tx_2y_2 + tx_3y_3 \\
&= t(x_1y_1 + x_2y_2 + x_3y_3) ,
\end{aligned}$$

which, by Theorem 8-3, is $t(\vec{X}\cdot\vec{Y})$.

Proof is complete.

Corollary. To prove that

$$\vec{X}\cdot(a\vec{Y} + b\vec{Z}) = a(\vec{X}\cdot\vec{Y}) + b(\vec{X}\cdot\vec{Z}) ,$$

we expand the left-hand member to obtain

$$\begin{aligned}
&[x_1,x_2,x_3]\cdot[ay_1 + bz_1 + ay_2 + bz_2 + ay_3 + bz_3] \\
&= ax_1y_1 + bx_1z_1 + ax_2y_2 + bx_2z_2 + ax_3y_3 + bx_3z_3] \\
&= a(x_1y_1 + x_2y_2 + x_3y_3) + b(x_1z_1 + x_2z_2 + x_3z_3) \\
&= a(\vec{X}\cdot\vec{Y}) + b(\vec{X}\cdot\vec{Z}) \text{, or the right-hand member.}
\end{aligned}$$

Hence, the proof is complete.

8-6 Although there are not many new ideas in this section, some of the arguments require close attention. The postulates and definition mentioned are from the SMSG Geometry.

We note that even though P and P_1 are in the plane M, $\overrightarrow{P} - \overrightarrow{P_1}$ denotes an origin-vector which does not lie in M unless M also contains the origin.

Example 3 through Example 6 are not essential to the development, but are included to show students that entire regions or their boundaries may be described concisely with vectors.

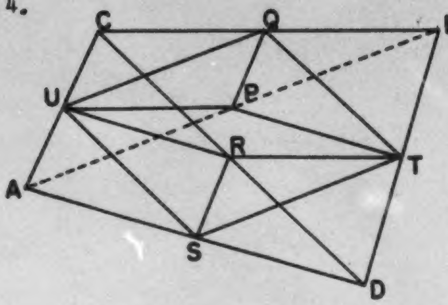
Exercises 8-6

1. (a) $7x - 3y + 5z = 15$

2. We assume the plane in question contains the given points in each of the following.
   (a) $2x - 3y + z - 14 = 0$
   (b) $2x - 4y + 7z + 69 = 0$
   (c) $3x - 5y + 4z - 50 = 0$
   (d) $x + y - 6z + 38 = 0$

3. (a) $\frac{5}{\sqrt{14}}$ or $\frac{5\sqrt{14}}{14}$
   (b) $\frac{8}{\sqrt{38}}$ or $\frac{4\sqrt{38}}{19}$
   (c) if $d = 0 : 0$
   if $d \neq 0 : \frac{|d|}{\sqrt{a^2 + b^2 + c^2}}$

4.



A,B,C, and D are the four vertices of the tetrahedron and P,Q,R,S,T and U are the midpoints of $\overline{AB}$,$\overline{BC}$,$\overline{CD}$,$\overline{DA}$,$\overline{BD}$, and $\overline{CA}$ respectively. Thus

$\vec{P} = \frac{1}{2}(\vec{A} + \vec{B})$ $\quad\vec{Q} = \frac{1}{2}(\vec{B} + \vec{C})$

$\vec{R} = \frac{1}{2}(\vec{C} + \vec{D})$ $\quad\vec{S} = \frac{1}{2}(\vec{D} + \vec{A})$

$\vec{T} = \frac{1}{2}(\vec{B} + \vec{D})$ $\quad\vec{U} = \frac{1}{2}(\vec{A} + \vec{C})$

(a) To show that $\overleftrightarrow{QS}$, $\overleftrightarrow{UT}$, and $\overleftrightarrow{PR}$ are concurrent. $\overleftrightarrow{QS}$ is represented by $x\vec{Q} + (1 - x)\vec{S}$ or $\frac{1}{2}((1 - x)\vec{A} + x\vec{B} + x\vec{C} + (1 - x)\vec{D})$. $\overleftrightarrow{UT}$ is represented by $y\vec{U} + (1 - y)\vec{T}$ or $\frac{1}{2}(y\vec{A} + (1 - y)\vec{B} + (1 - y)\vec{C} + y\vec{D})$. For these to intersect there must be x and y such that $(1 - x) = y$ and $(1 - y) = x$. But $x = y = \frac{1}{2}$ meets this condition so $\frac{1}{4}(\vec{A} + \vec{B} + \vec{C} + \vec{D})$ is on both $\overleftrightarrow{QS}$ and $\overleftrightarrow{UT}$. But

$$\frac{1}{4}(\vec{A} + \vec{B} + \vec{C} + \vec{D}) = \frac{1}{2}(\frac{1}{2}(\vec{A} + \vec{B}) + \frac{1}{2}(\vec{C} + \vec{D})) = \frac{1}{2}\vec{P} + (1 - \frac{1}{2})\vec{R}$$

which is on $\overleftrightarrow{PR}$. Hence the three lines are concurrent.

(b) We wish to show that QUST and PURT are parallelograms. First we must show that QUST and PURT are plane figures. However, from (a) we know that $\overleftrightarrow{QS}$ intersects $\overleftrightarrow{UT}$ and that $\overleftrightarrow{PR}$ intersects $\overleftrightarrow{UT}$, so we have coplanarity.

Then $\vec{Q} - \vec{U} = \frac{1}{2}(\vec{B} - \vec{A}) = \vec{T} - \vec{S}$, so $d(Q,U) = d(B,A)$,

and $\vec{Q} - \vec{T} = \frac{1}{2}(\vec{C} - \vec{D}) = \vec{U} - \vec{S}$, so $d(Q,T) = d(U,S)$.

Thus QUST has two pairs of opposite sides of equal length and is thus a parallelogram. We could not get from $\overleftrightarrow{QU} \| \overleftrightarrow{BA}$ and $\overleftrightarrow{TS} \| \overleftrightarrow{BA}$ to $\overleftrightarrow{QU} \| \overleftrightarrow{TS}$ without assuming or proving the theorem in solid geometry that for any lines a , b , and c , $a \| b$ and $b \| c$ imply $a \| c$. The proof that PURT is a parallelogram procedes similarly. $\overleftrightarrow{PR}$ intersects $\overleftrightarrow{UT}$, so P , U , R , and T are coplanar. We show that $\vec{P} - \vec{T} = \frac{1}{2}(\vec{A} - \vec{D}) = \vec{U} - \vec{R}$

and $\vec{P} - \vec{U} = \frac{1}{2}(\vec{B} - \vec{C}) = \vec{T} - \vec{R}$,

so that $d(P,T) = d(U,R)$ and $d(P,U) = d(R,T)$.

Hence PURT is a parallelogram.

(c) Since in (a) we show that $x = y = \frac{1}{2}$, the point of concurrency is the midpoint of the segments involved.

5. Let $\vec{D}$ be the normal vector from P_1 to M.

$$\vec{D} = [x - x_1, y - y_1, z - z_1] .$$

The unit vector normal to M is

$$\vec{n} = [\lambda, \mu, \nu] .$$

Then the distance from P_1 to M is found from

$$d = |\vec{n} \cdot \vec{D}| = |[\lambda,\mu,\nu] \cdot [x - x_1, y - y_1, z - z_1]|$$

$$= |\lambda x - \lambda x_1 + \mu y - \mu y_1 + \nu z - \nu z_1| = |\lambda x_1 + \mu y_1 + \nu z_1 - p| .$$

6. (a) $\overleftrightarrow{AB} = \{x : \vec{x} = [4, 9p - 7, -2p + 5]\}$

(b) $\overline{AB} = \{x : \vec{x} = [3 - 5p, 4 - p, 2 + p], 0 \leq p \leq 1\}$

(c) $\overrightarrow{AB} = \{x : \vec{x} = [3 - 5p, 4 - p, 2 + p], p \geq 0\}$

(d) $\overrightarrow{BA} = \{x : \vec{x} = [3 - 5p, 4 - p, 2 + p], p \leq 0\}$.

7. (a) Midpoint $\vec{M} = [3, 6, 7\frac{1}{2}]$

trisection points $\vec{T}_1 = [2,4,5]$ and $\vec{T}_2 = [4,8,10]$.

(b) $\vec{M} = [3\frac{1}{2}, -4\frac{1}{2}, 13]$

$\vec{T}_1 = [2\frac{1}{3}, -2\frac{1}{3}, 8\frac{2}{3}]$ and $\vec{T}_2 = [5\frac{2}{3}, -6\frac{2}{3}, 15]$.

(c) $\vec{M} = \frac{1}{2}[a_1 + b_1, a_2 + b_2, a_3 + b_3]$

$\vec{T}_1 = \frac{1}{3}[2a_1 + b_1, 2a_2 + b_2, 2a_3 + b_3]$ and

$\vec{T}_2 = \frac{1}{3}[a_1 + 2b_1, a_2 + 2b_2, a_3 + 2b_3]$.

8. (a) $\vec{X} = [0,0,0]$

(b) $\vec{X} = [\frac{2}{3}, 2\frac{1}{2}, -5\frac{1}{2}]$

(c) $\vec{X} = [1\frac{1}{4}, -\frac{3}{4}, 3\frac{1}{4}]$

9. (a) The triangular region is $\{Y : \overrightarrow{Y} = [1 + p - 2q + 2pq,$ $4 - p - 2q + 2pq, -2 + 3p + 6q - 6pq]$ where $0 \leq p \leq 1$ and $0 \leq q \leq 1\}$.

The interior is the same except $0 < p < 1$ and $0 < q < 1$.

The triangle is $\{Y : \overrightarrow{Y} = [1 + p - 2q + 2pq , 4 - p - 2q + 2pq ,$ $-2 + 3p + 6q - 6pq]$ where $(p = 0$ and $0 \leq q \leq 1)$ or $(q = 0$ and $0 \leq p \leq 1)$ or $(q = 1$ and $0 \leq p \leq 1)\}$.

(b) $p = q = \frac{1}{2}$ in the above gives the desired point $[1,3,1]$. Hence $[1,3,1]$ must be an interior point.

(c) If $[-4,-5,-6]$ is in the triangular region, then $p - 2q + 2pq = -4$ and $4 - p - 2q + 2pq = -5$. If we solve this, we find $p = 2$ and hence $[-4,-5,-6]$ cannot be in the triangular region.

Review Exercises

1. $z = 5$

2. $x^2 = 25$

3. $x^2 = z^2$

4. $y^2 + z^2 = 4$

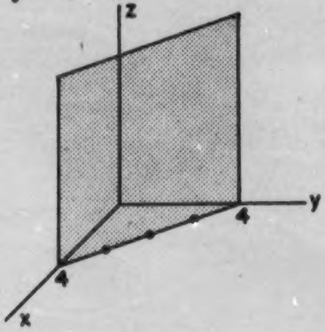
5. $x^2 + y^2 + z^2 = a^2$

6. $(x - 2)^2 + (y + 1)^2 + z^2 = r^2$

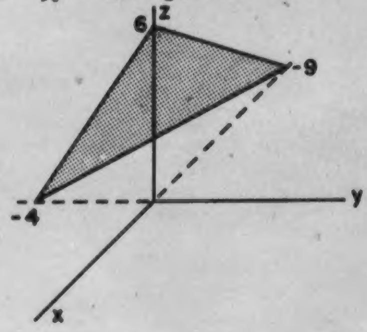
7. $(x - 1)^2 + (y - 2)^2 - 4z + 5 = 0$

8. $x + y + z - 6 = 0$

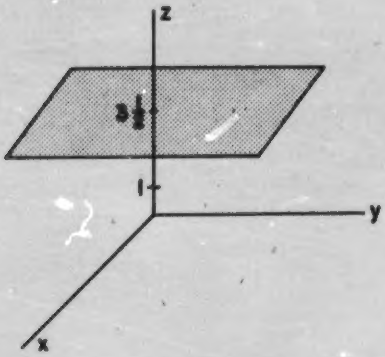
9. $x + y - 4 = 0$



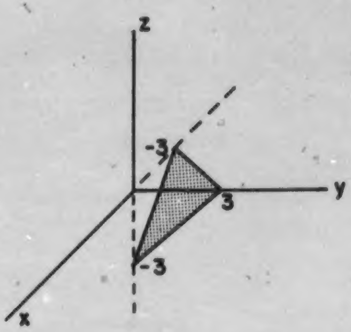
11. $4x + 9y - 6z + 36 = 0$



10. $2z - 7 = 0$



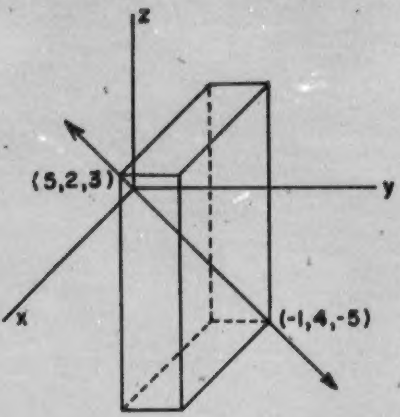
12. $x - y + z + 3 = 0$



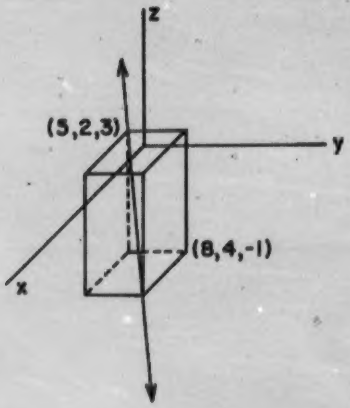
13. $x = 5 - 3t$

$y = 2 + t$

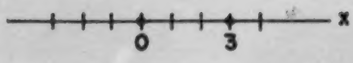
$z = 3 - 4t$

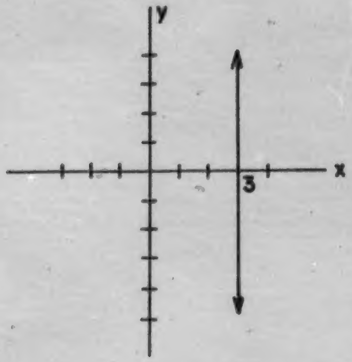


14. $\frac{x - 5}{-3} = \frac{y - 2}{-2} = \frac{z - 3}{4}$.



15.



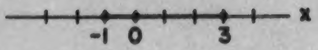


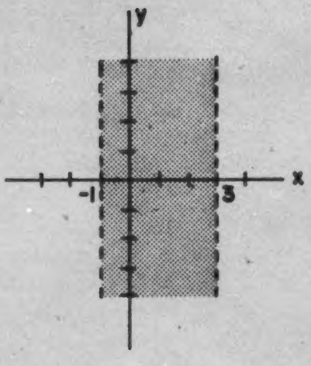
One-space: a point 3 units to the right of the origin.

2-space: a line perpendicular to the x-axis and 3 units to the right of the origin.

3-space: a plane perpendicular to the x-axis and 3 units in the positive direction from the origin.

16.



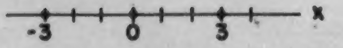


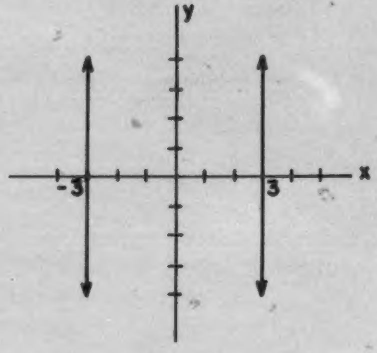
One space: a segment between, but not including, the points $x = -1$ and $x = 3$.

2-space: a portion of the xy-plane between, but not including, the lines $x = -1$ and $x = 3$.

3-space: a portion of space between, but not including, the planes $x = -1$ and $x = 3$. (It may be visualized as the path made by moving a plane parallel to the yz-plane.)

17.



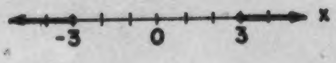


One-space: two points, $x = 3$ and $x = -3$.

2-space: two lines, $x = 3$ and $x = -3$.

3-space: two planes, $x = 3$ and $x = -3$.

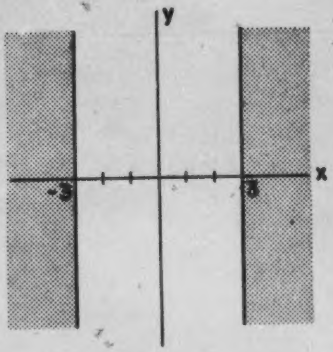
18.



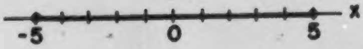
One-space: the portion of the x-axis to the right of and including $x = 3$, and to the left of and including $x = -3$.

2-space: the portion of the plane to the right of and including the line $x = 3$, and to the left of and including $x = -3$.

3-space: the portion of space beyond (in the positive direction) and including the plane $x = 3$, and the portion beyond (in the negative direction) and including the plane $x = -3$.



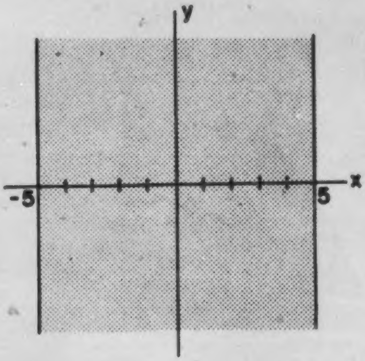
19.



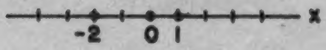
One-space: a segment between and including the points $x = -5$ and $x = 5$.

2-space: a portion of the xy-plane between and including the lines $x = -5$ and $x = 5$.

3-space: a portion of space between and including the planes $x = -5$ and $x = 5$.



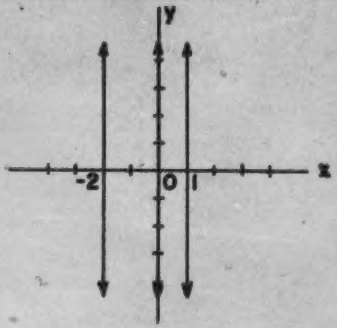
20.



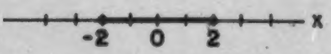
One-space: the points $x = -2$, $x = 0$, and $x = 1$.

2-space: the lines $x = -2$, $x = 0$, and $x = 1$.

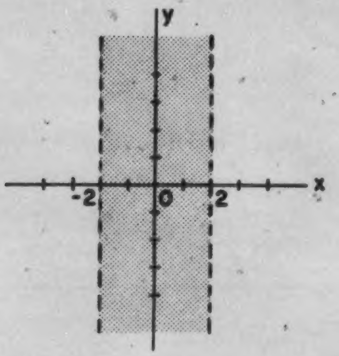
3-space: the planes $x = -2$, $x = 0$, and $x = 1$.



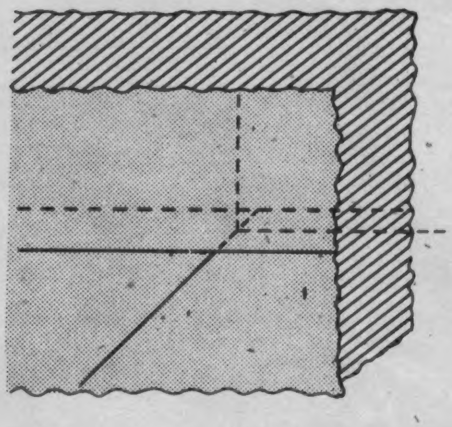
21. $R_1 = \{(x,y) : |x| < 2\}$



One-space: a segment between but excluding the points $x = -2$ and $x = 2$.

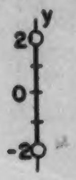


2-space: a portion of space between but excluding the lines $x = -2$ and $x = 2$.

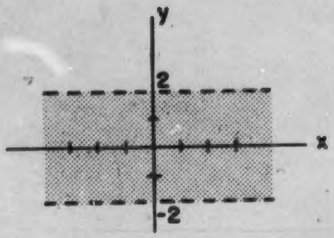


3-space: a portion of space between but excluding the planes $x = -2$ and $x = 2$.

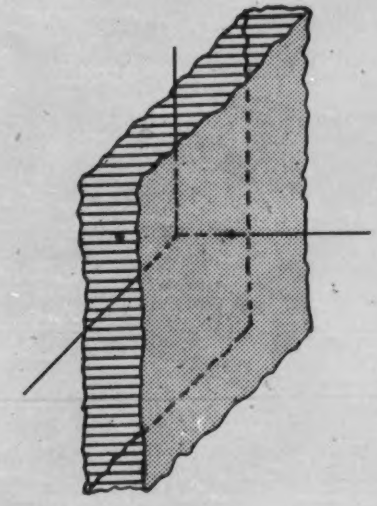
$R_2 = \{(x,y) : |y| < 2\}$



One-space (if we choose the y-axis):
a segment between but
excluding the points
$y = -2$ and $y = 2$.



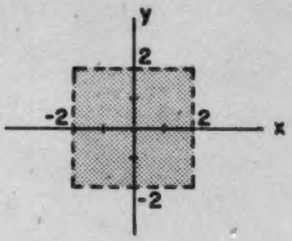
2-space: a portion of the plane
between but excluding the
lines $y = -2$ and $y = 2$.



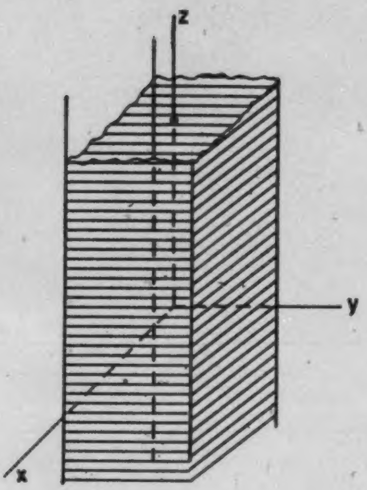
3-space: a portion of space between
but excluding the planes
$y = -2$ and $y = 2$.

$R_3 = R_1 \cap R_2$

One-space: R_3 is the null set



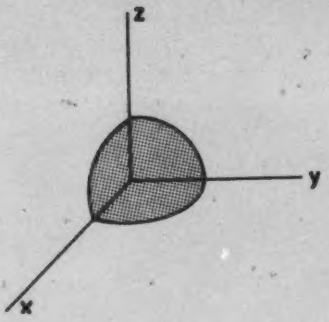
2-space: the interior of the square bounded by $x = -2$, $x = 2$ $y = -2$, $y = 2$.



3-space: the interior of the prism bounded by the plane $x = -2$, $x = 2$, $y = -2$, $y = 2$.

22. If, in Exercise 21, $<$ is changed to $\leq$ the graphs include the points, lines, and planes which are only boundaries and are not included in Exercise 21. $R_1 \cup R_2$ is the union of the first two graphs and would include all points in R_1 or R_2 .

23.



The graph of $x^2 + y^2 + z^2 \leq 1$ represents a sphere with the center at the origin and all the points within the sphere. (Only a portion of the graph is shown.) If $\leq$ is changed to $<$, the graph includes only the points within the sphere.

24.

	Distance	A	B	C	D	O
(a)	M_1	0	$\frac{5}{\sqrt{14}}$	$\frac{2}{\sqrt{14}}$	$\frac{17}{\sqrt{14}}$	$\frac{4}{\sqrt{14}}$
(b)	M_2	$\frac{4}{\sqrt{14}}$	$\frac{1}{\sqrt{14}}$	$\frac{2}{\sqrt{14}}$	$\frac{7}{\sqrt{14}}$	$\frac{3}{\sqrt{14}}$
(c)	M_3	$\frac{7}{\sqrt{14}}$	$\frac{13}{\sqrt{14}}$	$\frac{13}{\sqrt{14}}$	$\frac{9}{\sqrt{14}}$	$\frac{2}{\sqrt{14}}$
(d)	M_4	$\frac{5}{\sqrt{3}}$	$\frac{5}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{3}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$

25. (a) $\frac{x + 11}{5} = \frac{y}{1} = \frac{z - 18}{-7}$

(b) $\frac{x}{1} = \frac{y - 2}{1} = \frac{z - 2}{1}$

(c) $\frac{x - 1}{4} = \frac{y - 2}{3} = \frac{z}{1}$

(d) $\frac{x}{-1} = \frac{y - 5}{11} = \frac{z - 4}{7}$

(e) $\frac{x - 2}{3} = \frac{y - 3}{5} = \frac{z}{-2}$

(f) $\frac{x - 2}{5} = \frac{y - 1}{2} = \frac{z - 2}{3}$

26. (a) $[x,y,z] = [-11,0,18] + t[5,1,-7]$

(b) $[x,y,z] = [0,2,2] + t[1,1,1]$

(c) $[x,y,z] = [1,2,0] + t[4,-3,1]$

(d) $[x,y,z] = [0,5,4] + t[-1,11,7]$

(e) $[x,y,z] = [2,3,0] + t[3,5,-2]$

(f) $[x,y,z] = [2,1,2] + t[5,2,3]$

27. One method is to find direction numbers; for $\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$ direction numbers are $(-2,2,6)$, hence they are parallel. Direction numbers for $\overleftrightarrow{BC}$ and $\overleftrightarrow{AD}$ are $(5,-4,-4)$.

28. Parametric equations of the medians

from A, $x = -t_1$, $y = 3t_1$, $z = -t_1$.

from B, $x = 2 - 4t_2$, $y = 4 - 3t_2$, $z = 6 - 10t_2$.

from C, $x = -4 + 5t_3$, $y = 2$, $z = -8 + 11t_3$.

The medians are concurrent in the point $(-\frac{2}{3}, 2, -\frac{2}{3})$.

29. $a = 7$

30. $a = -3$

Teachers' Commentary

Chapter 9

QUADRIC SURFACES

Since many of the students who study this course are likely candidates for college-level mathematics, this chapter has been included to give the students help in vizualizing and handling the types of objects they will encounter in later courses. Our sights are particularly set on the calculus. Even in the elementary applications of calculus, one encounters solid, or 3-space, figures of the non-rectangular variety. It might aid some students if you were to roughly describe calculus as a sort of super-algebra which enables us to handle areas of objects with curved sides and volumes of objects with curved surfaces. Ordinary algebra is generally powerless with objects which do not have straight sides or flat surfaces. Of course, this would not be a complete description of the power of the calculus, but some such discussion could be used to motivate the study of this chapter.

Except for the Challenge Problems and an occasional natural extension in the Exercises, we have limited the discussion to very simple forms of surfaces. In most cases the origin is chosen as the center of the figure, and the axes or elements of the figure are oriented along some coordinate axis. This simplifies the drawing techniques and the algebraic manipulations. More complicated forms are obtained by simple extensions.

Quadrics hold much the same place among surfaces that the conics occupy among curves and, next to the plane, are by far the most important types of surfaces.

It is usual to identify nine species of quadric surfaces, but in order to keep life in this chapter simple we have presented only six of these in the students' text. The other three types (numbers 5, 7, and 8 in the list below) appear in Challenge Problems. For completeness we list the nine species with an example equation for each.

1. Ellipsoid. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

2. Hyperboloid of one sheet. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$.

3. Hyperboloid of two sheets. $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$.

4. Elliptic paraboloid. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - cz = 0$.

5. Hyperbolic paraboloid. $\frac{x^2}{a^2} - \frac{y^2}{b^2} - cz = 0$.

6. Elliptic cylinder. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

7. Parabolic cylinder. $x^2 - cz = 0$.

8. Hyperbolic cylinder. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

9. Elliptic cone. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$.

It is of considerable interest to note how the surface changes when the equation is altered by changing a sign, or by changing the power of a variable, or by changing the value of one or more of the constants. However, we felt that this material would make the course too long, so we somewhat reluctantly restricted the quadrics to the six simplest ones.

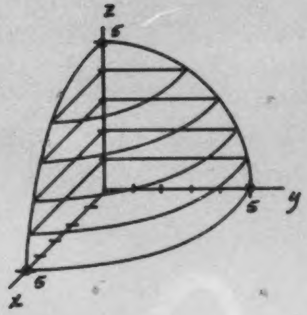
For the sake of variety, some of the spheres in Exercises 9-2 have been located away from the origin. If the coefficients of x^2 , y^2 , and z^2 are all equal, then the quadratic represents a sphere. One may then complete the square for each variable and obtain an equation of the form

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2 .$$

The point (x_0,y_0,z_0) is the center and r is the radius.

Exercises 9-2

1.



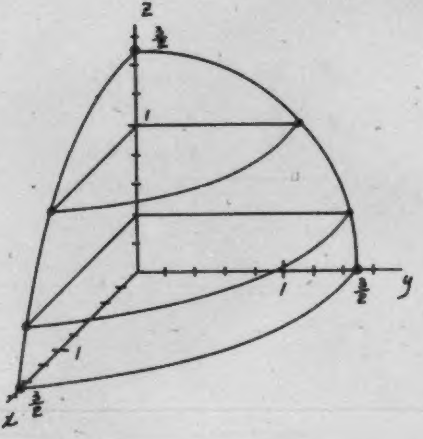
A sphere.

Center: origin

Radius: 5

All traces are circles.

2.


z
$\frac{3}{2}$
1
1
$\frac{3}{2}$
y
1
$\frac{3}{2}$
x

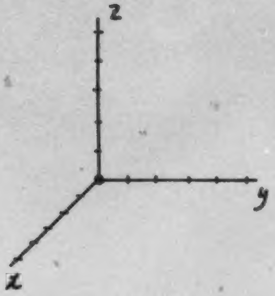
A sphere.

Center: origin

Radius: $\frac{3}{2}$

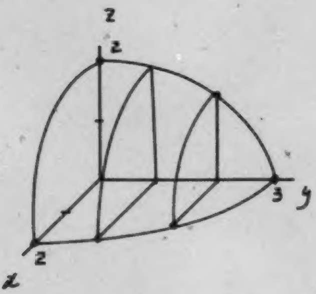
All traces are circles.

3.



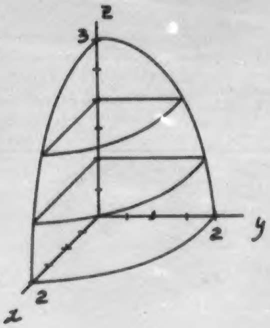
A point-sphere.
Center: origin
Radius: 0
The origin is the only point of the locus.

4.



A prolate spheroid.
Center: origin
The xy- and yz-traces are ellipses.
The xz-trace is a circle.

5.



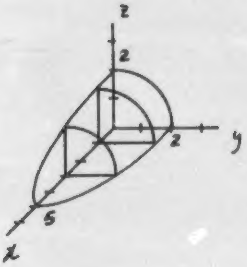
A prolate spheroid.

Center: origin

The xz- and yz-traces are ellipses.

The xy-trace is a circle.

6.



A prolate spheroid.

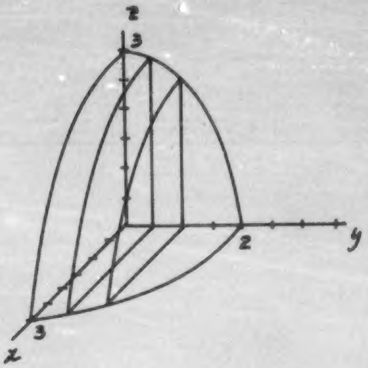
Center: origin

The xy- and xz-traces are ellipses.

The yz-trace is a circle.

164

7.



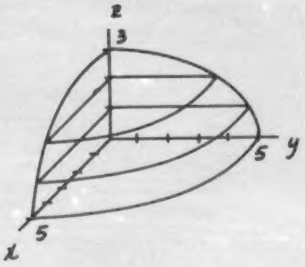
An oblate spheroid.

Center: origin

The xy- and yz-traces are ellipses.

The xz-trace is a circle.

8.



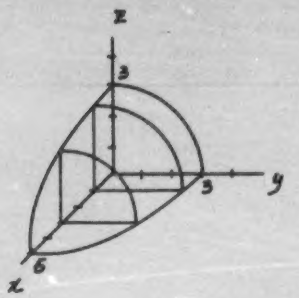
An oblate spheroid.

Center: origin

The xz- and yz-traces are ellipses.

The xy-trace is a circle.

9.



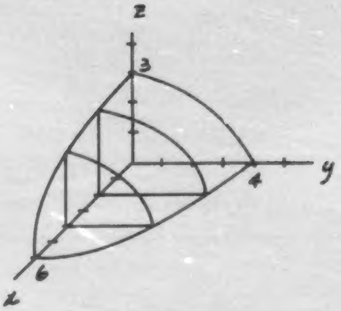
A prolate spheroid.

Center: origin

The xy- and xz-traces are ellipses.

The yz-trace is a circle.

10.

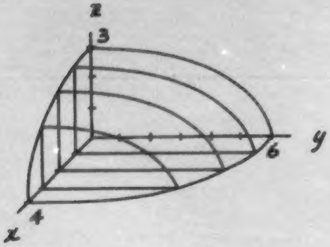


An ellipsoid

Center: origin

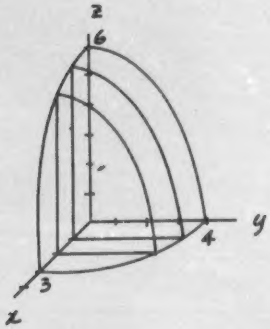
All traces are ellipses.

11.



An ellipsoid.
Center: origin
All traces are ellipses.

12.



An ellipsoid.
Center: origin
All traces are ellipses.

13. $\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} = r$,

or $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$.

14. $x^2 - 2xx_0 + x_0{}^2 + y^2 - 2yy_0 + y_0{}^2 + z^2 - 2zz_0 + z_0{}^2 = r^2$,

or $x^2 + y^2 + z^2 - (2x_0)x - (2y_0)y - (2z_0)z + (x_0{}^2 + y_0{}^2 + z_0{}^2 - r^2) = 0$.

Since (x_0,y_0,z_0) represents any point and $r > 0$, the given equation represents a sphere with radius r and center at (x_0,y_0,z_0) .

15. (a) $x^2 + y^2 + z^2 - 4x - 2y - 6z - 11 = 0$.

(b) $x^2 + y^2 + z^2 + 2y - 4z + 1 = 0$.

(c) $x^2 + y^2 + z^2 - 2x - 6y + 4z + 12 = 0$.

(d) $x^2 + y^2 + z^2 - \frac{2}{3}x + 2y - z + \frac{13}{36} = 0$,

or $36x^2 + 36y^2 + 36z^2 - 24x + 72y - 36z + 13 = 0$.

(e) $x^2 + y^2 + z^2 - x - \frac{1}{2}y + z + \frac{5}{16} = 0$,

or $16x^2 + 16y^2 + 16z^2 - 16x - 8y + 16z + 5 = 0$.

(f) $x^2 + y^2 + z^2 - 3x + y - 5z - .25 = 0$

or $4x^2 + 4y^2 + 4z^2 - 12x + 4y - 20z - 1 = 0$.

16. (a) Center: origin. Radius: $\sqrt{3}$.
(b) Center: $(1,-2,3)$. Radius: 2 .
(c) Center: $(0,2,-1)$. Radius: 5 .
(d) Center: $(-3,4,-7)$. Radius: $\sqrt{2}$.
(e) Center: $(-2,3,0)$. Radius: 0 . (a point-sphere.)
(f) Not a sphere.
(g) Center: $(\frac{1}{2},\frac{2}{3},-1)$. Radius: $\frac{1}{2}$.
(h) Center: $(\frac{3}{4}, 2, -\frac{1}{2})$. Radius: $\frac{3}{2}$.

17. Center: $(0,1,5)$. Radius: $\sqrt{6}$

Equation of the sphere: $x^2 + y^2 + z^2 - 2y - 10z + 20 = 0$.

18. $\frac{x^2}{9} + \frac{y^2}{49} + \frac{z^2}{25} = 1$, or $1225x^2 + 225y^2 + 441z^2 = 11025$.

Challenge Problems

1. $\frac{(x-3)^2}{36} + \frac{(y+1)^2}{16} + \frac{(z-2)^2}{144} = 1$,

   or $4x^2 + 9y^2 + z^2 - 24x + 18y - 4z - 95 = 0$.

2. Substituting the coordinates of the four points into the equation of general form, $x^2 + y^2 + z^2 + Dx + Ey + Fz + G = 0$, results in

$$\begin{array}{ll}
(1) & 3E + F + G = -10 \,, \\
(2) & -2D + 2F + G = -8 \,, \\
(3) & D + E + 4F + G = -18 \,, \\
(4) & -3D + 3E + 2F + G = -22 \,.
\end{array}$$

From these we obtain

$$\begin{array}{lll}
 & (2) & -2D + 2F + G = -8 \,, \\
(5) & (1) - (4) & 3D - F = 12 \,, \\
(6) & 3 \cdot (3) - (1) & 3D + 11F + 2G = -44 \,.
\end{array}$$

Then we have

$$\begin{array}{lll}
(7) & (6) - 2 \cdot (2) & 7D + 7F = -28 \,, \\
(8) & 7 \cdot (5) & 21D - 7F = 84 \,, \text{ and} \\
(9) & (7) + (8) & 28D = 56 \,.
\end{array}$$

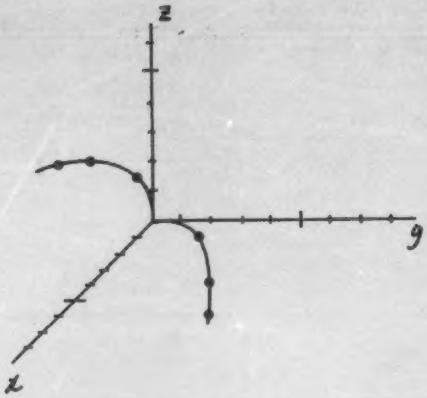
Therefore, $D = 2$, $F = -6$, $G = 8$, $E = -4$, and the equation is $x^2 + y^2 + z^2 + 2x - 4y - 6z + 8 = 0$. This can be written $(x+1)^2 + (y-2)^2 + (z-3)^2 = 6$, showing the center and radius of the sphere.

As an alternate method, find the center of the sphere as the intersection of the perpendicular bisecting planes determined by pairs of the given points. The radius may then be found as the distance between the center and one of the given points.

Four points determine a sphere if the points are not coplanar and if no three points are collinear.

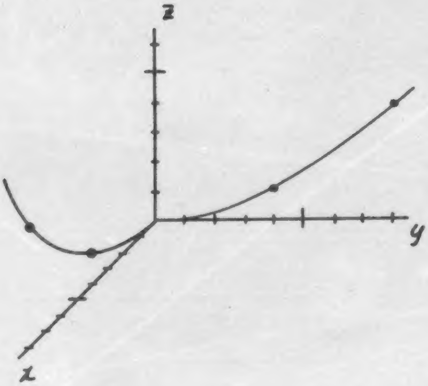
Exercises 9-3

1.



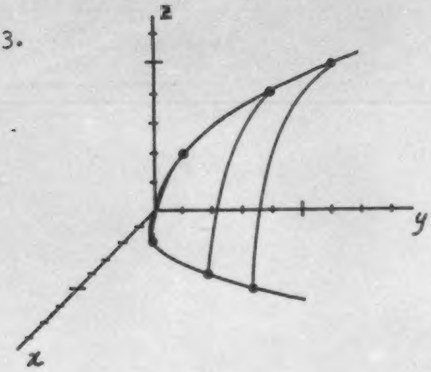
A paraboloid of revolution.
Axis: x-axis
Vertex: origin
The xy- and xz-traces are parabolas. Sections parallel to the yz-plane are circles. (These sections are not drawn because they interfere with other parts of the figure.)

2.



A paraboloid of revolution.
Axis: z-axis
Vertex: origin
The xz- and yz-traces are parabolas. Sections parallel to the xy-plane are circles (not shown).

3.



A paraboloid of revolution.

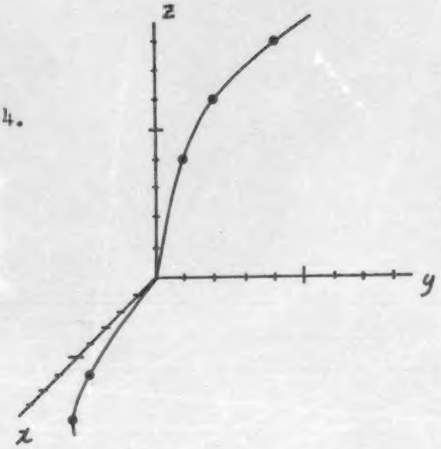
Axis: y-axis

Vertex: origin

The xy- and yz-traces are parabolas.

Sections parallel to the xz-plane are circles.

4.



An elliptic paraboloid.

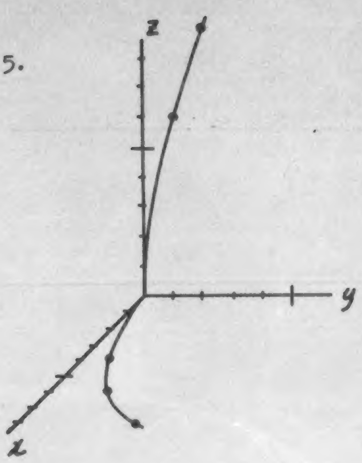
Axis: y-axis

Vertex: origin

The xy- and yz-traces are parabolas.

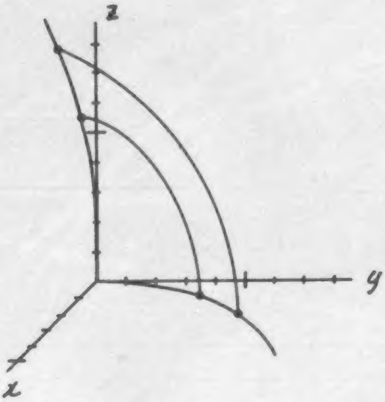
Sections parallel to the xz-plane are ellipses (not shown).

5.



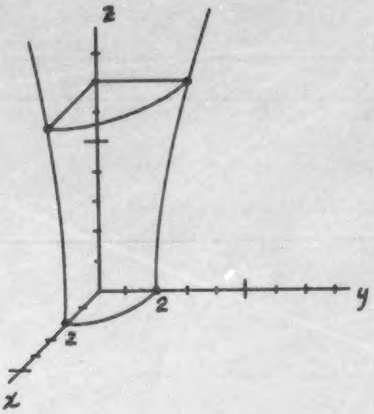
An elliptic paraboloid.
Axis: y-axis
Vertex: origin
The xy- and yz-traces are parabolas.
Sections parallel to the xz-plane
are ellipses (not shown).

6.



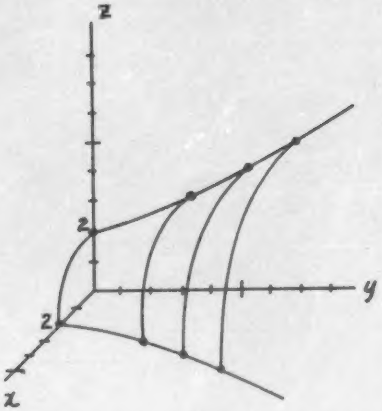
An elliptic paraboloid.
Axis: x-axis
Vertex: origin
The xy- and xz-traces are parabolas.
Sections parallel to the yz-plane
are ellipses.

7.



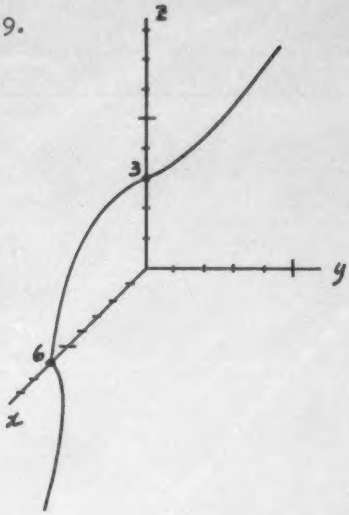
A hyperboloid of revolution (one sheet).
Axis: z-axis
The xy-trace is a circle of radius 2 .
The xz- and yz-traces are hyperbolas.

8.



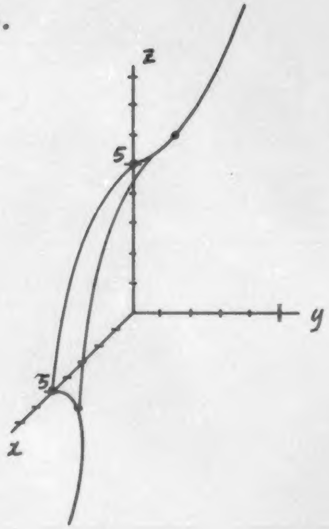
A hyperboloid of revolution (one sheet).
Axis: y-axis
The xz-trace is a circle of radius 2 .
The xy- and yz-traces are hyperbolas.

9.



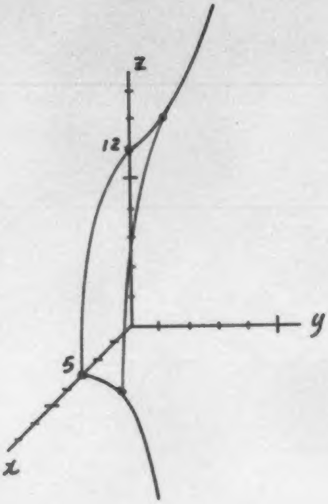
An elliptic hyperboloid (one sheet).
The xz-trace is an ellipse
The xy- and yz-traces are hyperbolas.

10.



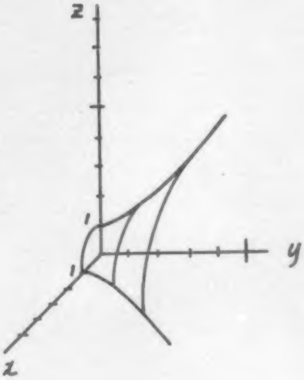
A hyperboloid of revolution (one sheet).
Axis: y-axis
The xz-trace is a circle of radius 5 .
The xy- and yz-traces are hyperbolas.

11.



An elliptic hyperboloid (one sheet).
The xz-trace is an ellipse.
The xy- and yz-traces are hyperbolas.

12.



A hyperboloid of revolution (one sheet).
Axis: y-axis
The xz-trace is a circle of radius 1 .
The xy- and yz-traces are hyperbolas.

13. In Section 7-7 we have $e = \frac{\sqrt{a^2 - b^2}}{a} < 1$ as the eccentricity of an ellipse. Applying this to the equation of the xy-trace of Equation (3), $\frac{x^2}{4} + \frac{y^2}{9} = 1$, we find $e = \frac{\sqrt{5}}{3}$. For any section of the hyperboloid parallel to the xy-plane, say when $z = k$, we have

$\frac{x^2}{4} + \frac{y^2}{9} = 1 + \frac{k^2}{25}$. Let $1 + \frac{k^2}{25} = q^2$, $q > 0$, the equation then becomes

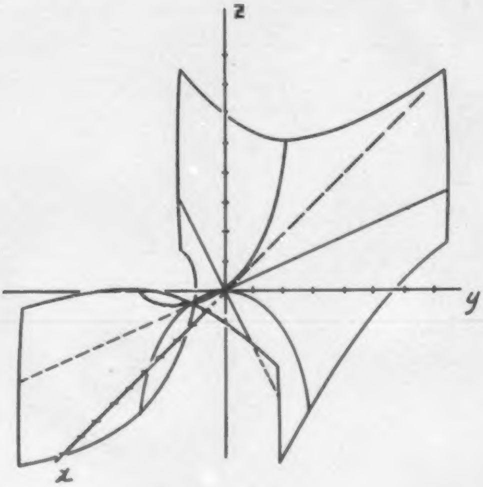
$\frac{x^2}{4q^2} + \frac{y^2}{9q^2} = 1$. Evaluating the eccentricity gives us

$$e = \frac{\sqrt{9q^2 - 4q^2}}{3q} = \frac{\sqrt{5}}{3} .$$

363 The Challenge Problems which follow contain work on hyperbolic paraboloids which were omitted from the basic text. An aid to identifying several of these surfaces is in their names. The first part of the name (an adjective) indicate the kind of sections we find parallel to one coordinate plane; the second word (a noun) indicates the type of sections which are parallel to the other two coordinate planes.

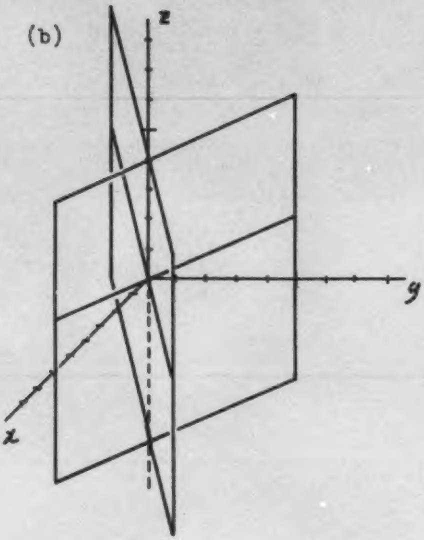
Challenge Problems

1. (a)



It will be noted that these "saddle shapes" are difficult to sketch.
The yz-trace and the sections parallel to it are parabolas opening downward. The xz-trace and the sections parallel to it are parabolas opening upward. The sections parallel to the xy-plane are hyperbolas, which degenerate to a pair of intersecting lines as the xy-trace.

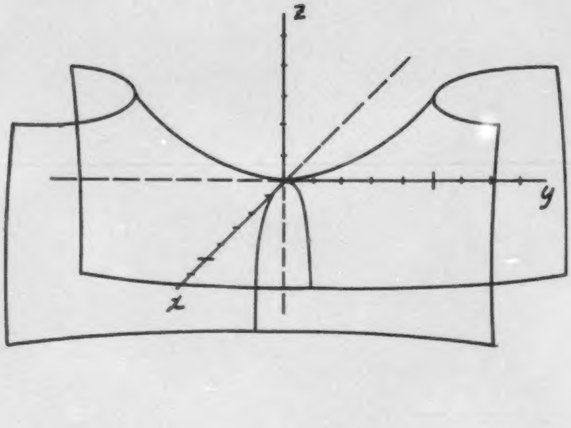
(b)



The planes shown in part (b) intersect on the z-axis and are determined by this axis and the xy-trace. These planes serve in an asymptotic capacity with respect to the hyperbolic paraboloid. A section parallel to the xy-plane (a hyperbola) will have as asymptotes the section of these two planes formed by the horizontal cutting plane.

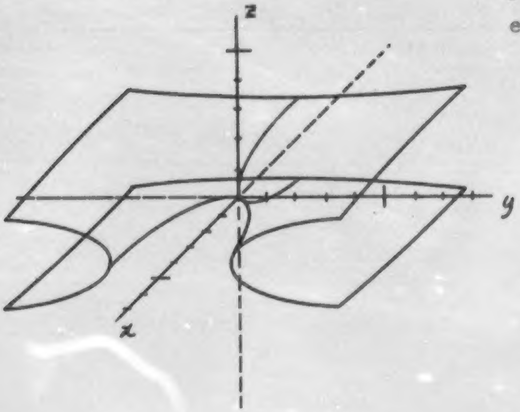
If students encounter difficulty vizualizing this surface, have them look at the region between two knuckles of a clenched fist.

2. The sections parallel to the xz-plane are parabolas opening downward. The sections parallel to the yz-plane are parabolas opening upward.
The sections parallel to the xy-plane are hyperbolas, except for the xy-trace which is a pair of intersecting lines.



177

3.

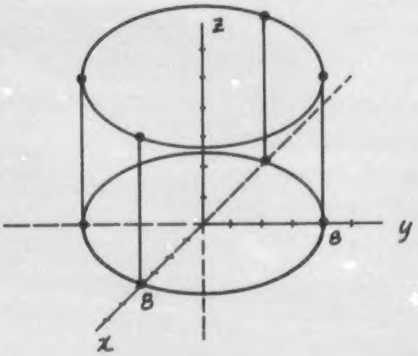


The sections parallel to the xy- and xz-planes are parabolas: the sections parallel to the yz-plane are hyperbolas, except for the degenerate yz-trace.

364 A cylinder need not be "round". Any curve (or line) may be a directrix. We have purposely included in the basic text only closed cylinders. Other types, including sinusoidal ones, are found in the Challenge Problems.

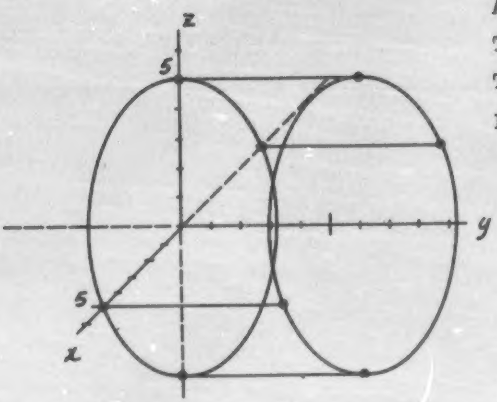
Exercises 9-4

1.



Axis of revolution: z-axis
The xy-trace and sections parallel to the xy-plane are circles of radius 8 .
The xz- and yz-traces are lines parallel to the z-axis.

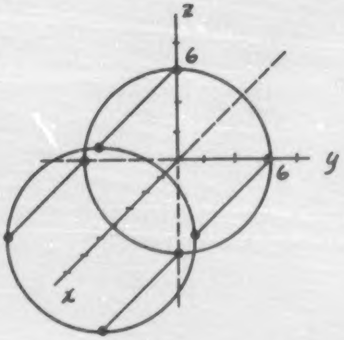
2.



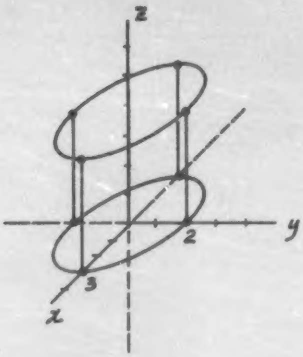
Axis of revolution: y-axis
The xz-trace is a circle of radius 5 .
The xy- and yz-traces are lines parallel to the y-axis.

3.

Axis of revolution: x-axis
The yz-trace is a circle of radius 6 .
The xy- and xz-traces are lines parallel to the x-axis.

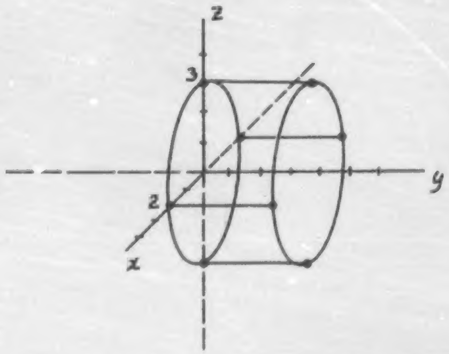


4.



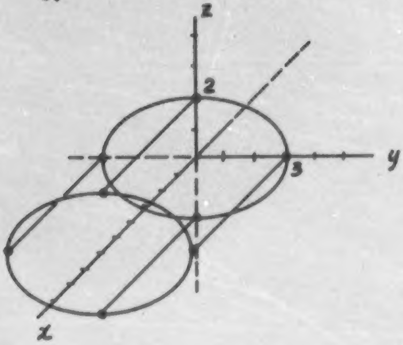
The xy-trace (and sections parallel to it) is an ellipse.
The xz- and yz-traces are lines parallel to the z-axis.

5.



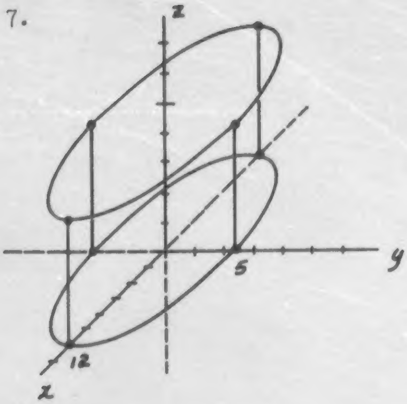
The xz-trace is an ellipse.
The xy- and yz-traces are lines parallel to the y-axis.

6.



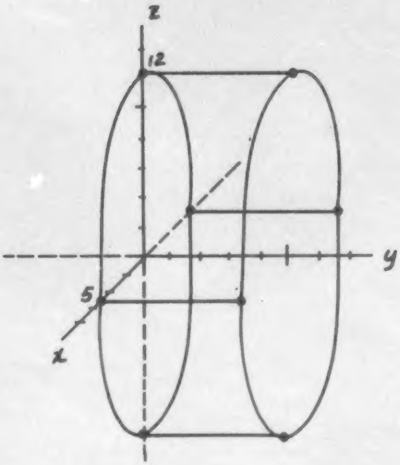
The yz-trace is an ellipse.
The xy- and yz-traces are lines
parallel to the x-axis.

7.



The xy-trace is an ellipse.
The xz- and yz-traces are lines
parallel to the z-axis.

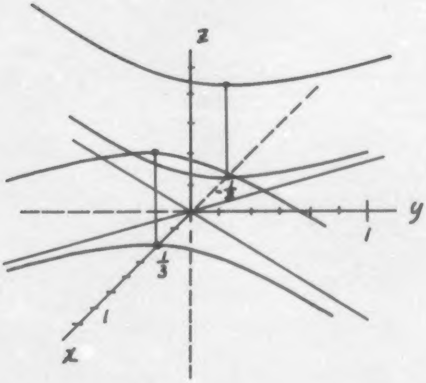
8.



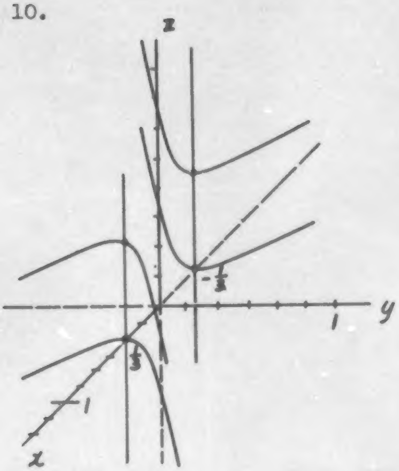
The xz-trace is an ellipse.
The xy- and yz-traces are lines parallel to the y-axis.

9.

The xy-trace (and sections parallel to it) is a hyperbola.
The xz-trace is a pair of lines parallel to the z-axis.
There is no yz-trace, but planes parallel to the yz-plane which intersect the surface cut off lines parallel to the z-axis.



10.



The xy-trace is a hyperbola.
The xz-trace is a pair of lines parallel to the z-axis.
There is no yz-trace, but planes parallel to the yz-plane which intersect the surface cut off lines parallel to the z-axis.

11. (a) $y^2 + z^2 = 81$.
    (b) $x^2 + z^2 = 36$.
    (c) $x^2 + y^2 = 16$.

12. (a) $y^2 + z^2 = 9$.
    (b) $x^2 + z^2 = 25$.
    (c) $x^2 + y^2 = 100$.

13. $x^2 + z^2 = 100$.

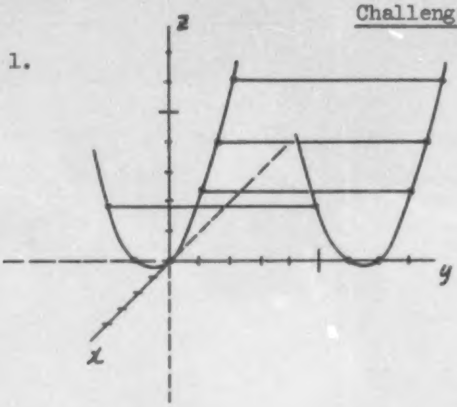
14. $y^2 + z^2 = 144$.

15. $x^2 + z^2 = 4$.

16. $25y^2 + 4z^2 = 100$.

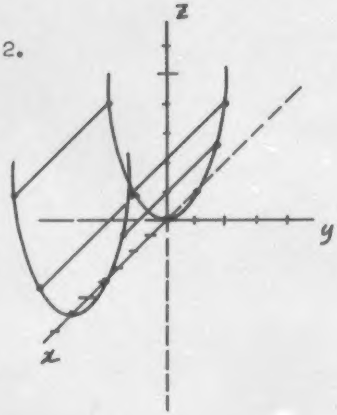
Challenge Problems

1.



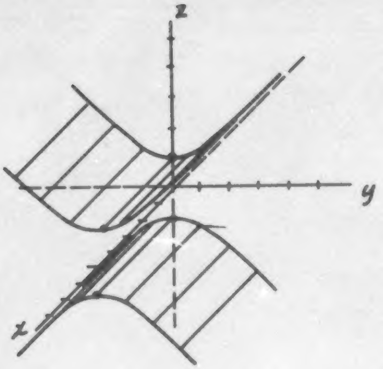
A parabolic cylinder.
The elements are parallel to the y-axis.
The directrix is a parabola.

2.



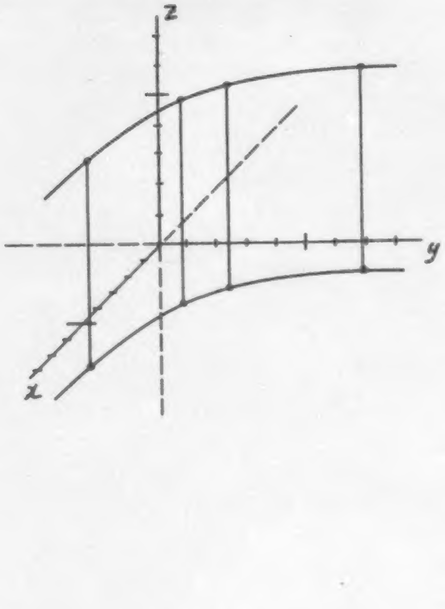
A parabolic cylinder.
The elements are parallel to the x-axis.
The directrix is a parabola.

3.



A hyperbolic cylinder.
The elements are parallel to the x-axis.
The directix is a hyperbola.

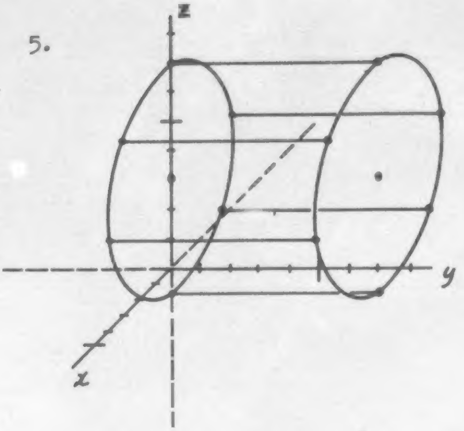
4.



A hyperbolic cylinder (only one branch shown).
The elements are parallel to the z-axis.
The directrix is an equilateral hyperbola.

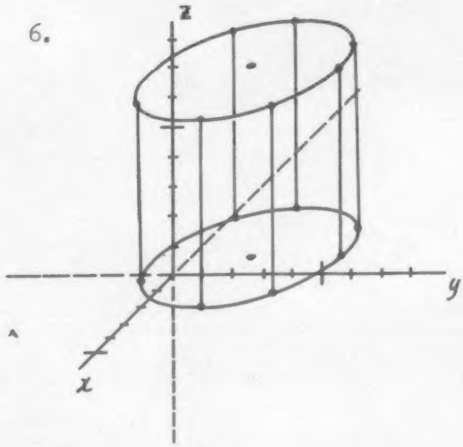
185

5.



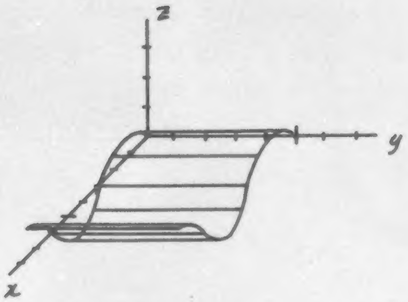
A circular cylinder.
The elements are parallel to the y-axis.
A directrix is a circle of radius 4 having its center at (0,0,3) .

6.



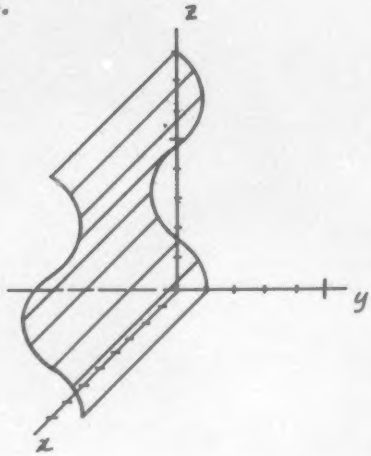
A circular cylinder.
The elements are parallel to the z-axis.
A directrix is a circle of radius 3 having its center at (-1,2,0) .

7.



The elements are parallel to the y-axis.
The directrix is a sine curve.

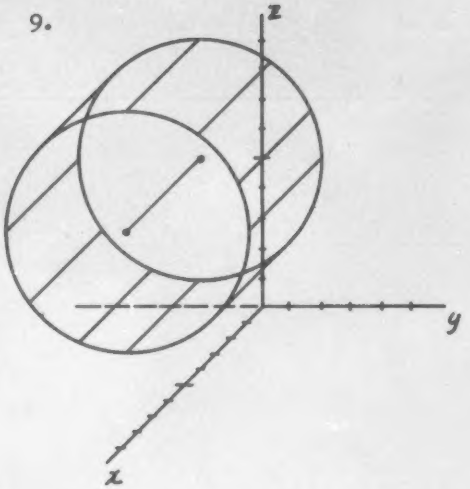
8.



The elements are parallel to the x-axis.
The directrix is a cosine (or displaced sine) curve.

187

9.

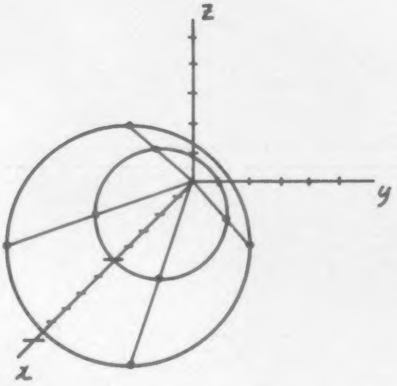


$(y + 2)^2 + (z - 5)^2 = 16$, or

$y^2 + z^2 + 4y - 10z + 13 = 0$.

Exercises 9-5

1.

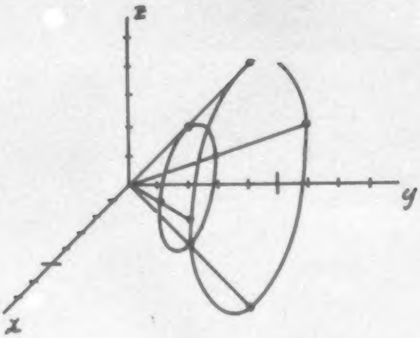


Axis: x-axis

Intercept: origin

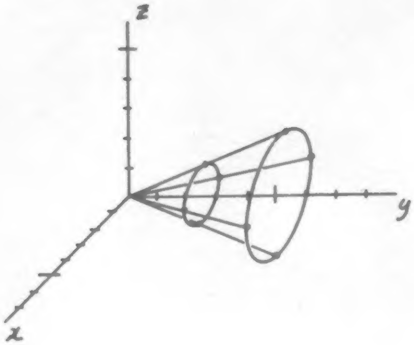
Sections parallel to the yz-plane are circles. (One nappe only is shown.)

2.



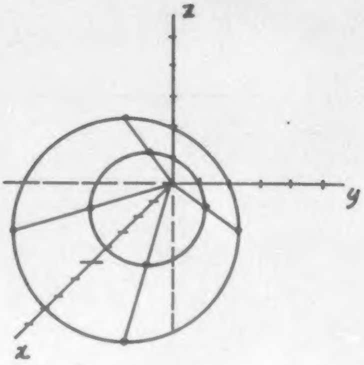
Axis: y-axis
Intercept: origin
Sections parallel to the xz-plane
are circles.

3.



Axis: y-axis
Intercept: origin
Sections parallel to the xz-plane
are circles.

4.

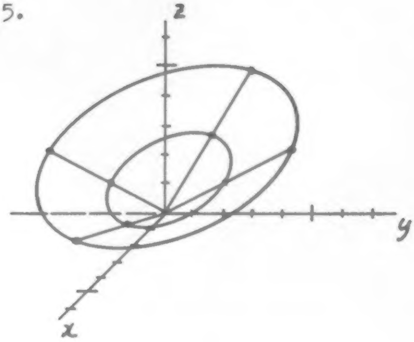


Axis: x-axis

Intercept: origin

Sections parallel to the yz-plane are circles.

5.

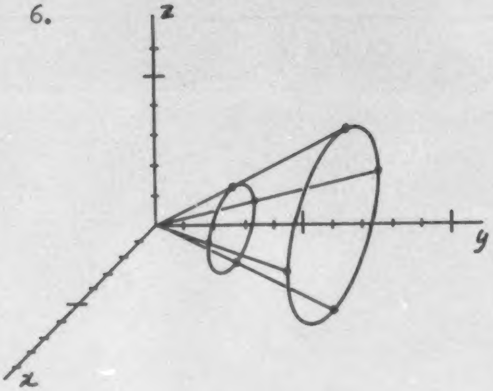


Axis: z-axis

Intercept: origin

Sections parallel to the xy-plane are ellipses.

6.



Axis: y-axis

Intercept: origin

Sections parallel to the xz-axis are ellipses.

7. $x^2 - 4y^2 + z^2 = 0$.

8. $-4x^2 + 4y^2 + 9z^2 = 0$.

9. $16x^2 + 16y^2 - 9z^2 = 0$.

10. $225x^2 - 16y^2 + 25z^2 = 0$.

11. The section in the plane $y = 1$ has the equation $\frac{x^2}{4} + \frac{z^2}{9} = 1$. The eccentricity of this ellipse is $e = \frac{\sqrt{9 - 4}}{3} = \frac{\sqrt{5}}{3}$. For any section of the cone parallel to the xz-plane, say when $y = k$, we have $\frac{x^2}{4} + \frac{z^2}{9} = k^2$, or $\frac{x^2}{4k^2} + \frac{z^2}{9k^2} = 1$. Evaluating the eccentricity gives us $e = \frac{\sqrt{9k^2 - 4k^2}}{3k} = \frac{\sqrt{5}}{3}$.

Challenge Problems

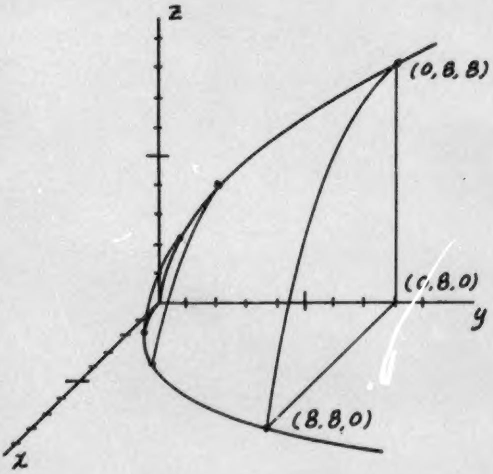
1. Since $a = 6$, $e = \frac{4}{6} = \frac{\sqrt{36 - 20}}{6}$; therefore, the ellipse in the plane $x = 1$ either has equation $\frac{y^2}{36} + \frac{z^2}{20} = 1$ or has equation $\frac{y^2}{20} + \frac{z^2}{36} = 1$. The cone is either $-180x^2 + 5y^2 + 9z^2 = 0$ or $-180x^2 + 9y^2 + 5z^2 = 0$.

2. Since $a = 8$, $e = \frac{4}{8} = \frac{\sqrt{64 - 48}}{8}$; therefore, the ellipse in the plane $z = 2$ either has equation $\frac{x^2}{16} + \frac{y^2}{12} = 1$ or has equation $\frac{x^2}{12} + \frac{y^2}{16} = 1$. The cone is either $4x^2 + 3y^2 - 48z^2 = 0$ or $3x^2 + 4y^2 - 48z^2 = 0$.

368 An interesting oral exercise might be interposed in this section. Have the students try to describe the surface generated by revolving about an axis of symmetry the printed capital form of certain letters of the English alphabet.

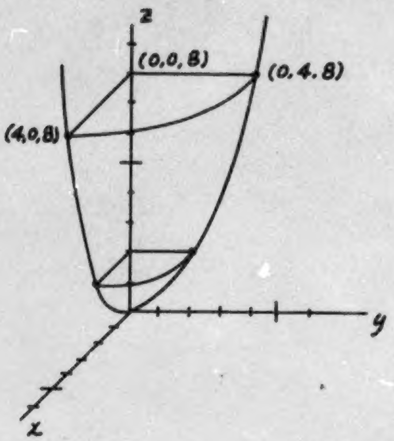
Exercises 9-6

1.



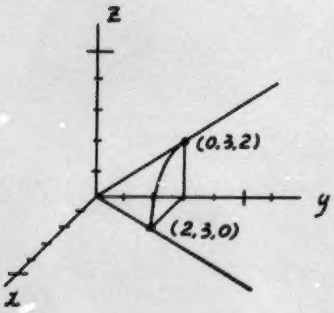
$x^2 + z^2 = 8y$.

2.


z
(0,0,8)
(0,4,8)
(4,0,8)
y
x

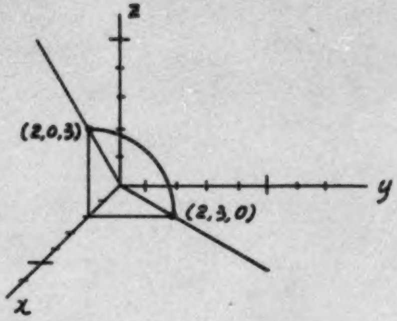
$x^2 + y^2 = 2z$.

3.



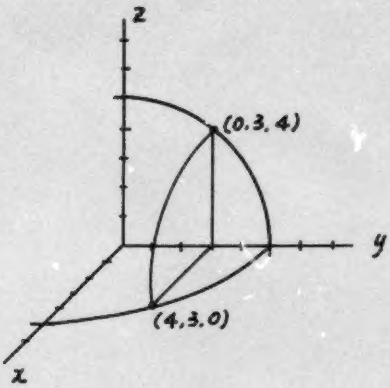
$9x^2 - 4y^2 + 9z^2 = 0$.

4.



$-9x^2 + 4y^2 + 4z^2 = 0$.

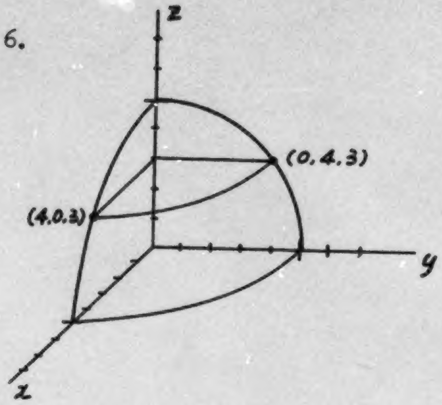
5.



$x^2 + y^2 + z^2 = 25$.

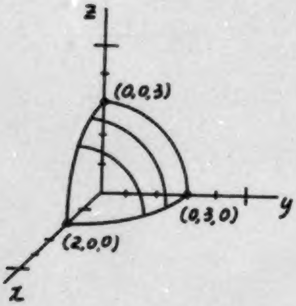
194

6.



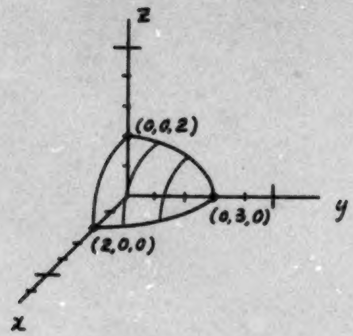
$x^2 + y^2 + z^2 = 25$.

7.


z
(0,0,3)
(0,3,0)
y
(2,0,0)
x

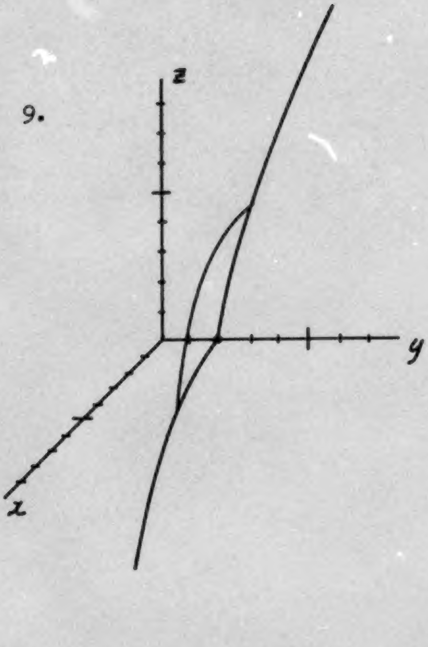
$9x^2 + 4y^2 + 4z^2 = 36$.

8.


z
(0,0,2)
(0,3,0)
y
(2,0,0)
x

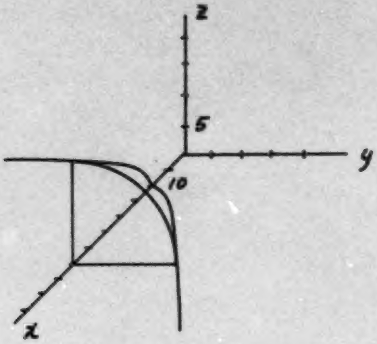
$9x^2 + 4y^2 + 9z^2 = 36$.

9.



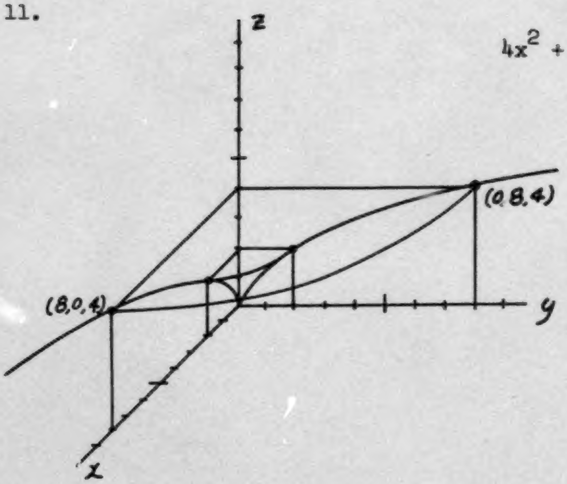
$x^2 - 4y^2 + z^2 + 16 = 0$.

10.



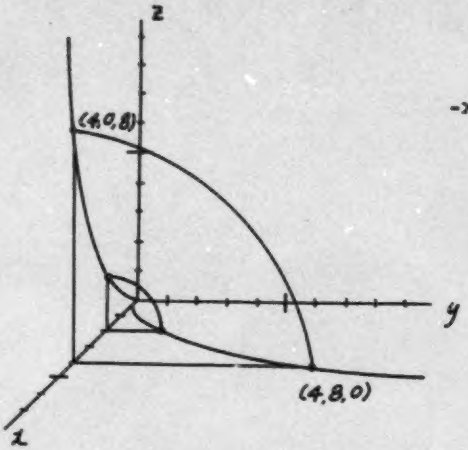
$-x^2 + 4y^2 + 4z^2 + 100 = 0$.

11.



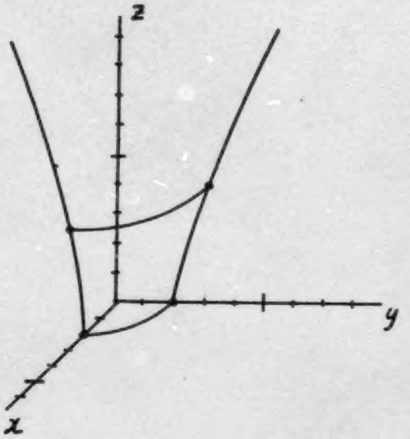
$4x^2 + 4y^2 - z^4 = 0$.

12.



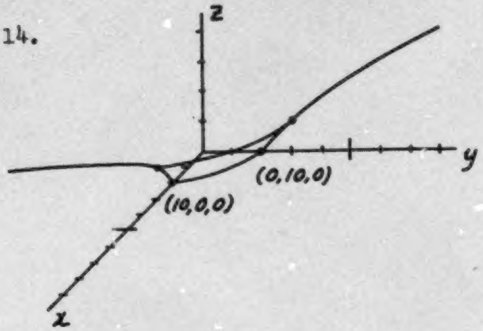
$-x^4 + 4y^2 + 4z^2 = 0$.

13.



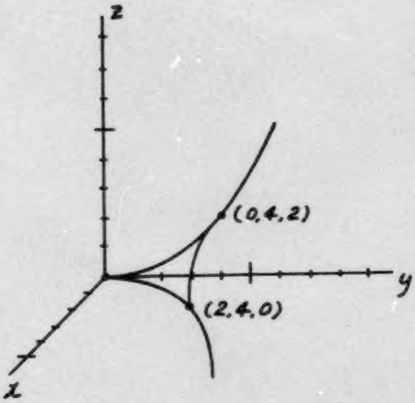
$4x^2 + 4y^2 - z^2 = 16$.

14.


z
y
x
(0,10,0)
(10,0,0)

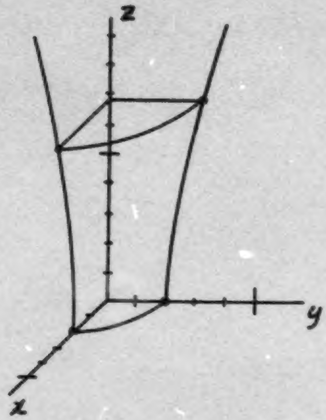
$x^2 + y^2 - 4z^2 = 100$.

15.



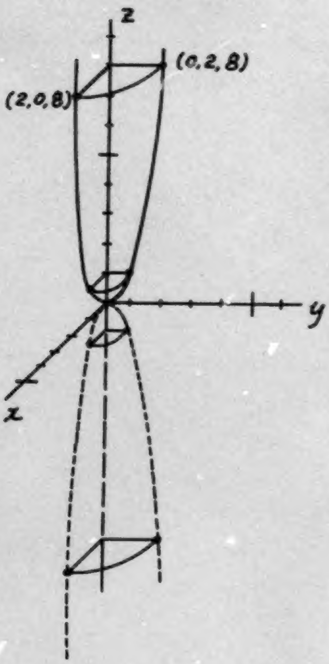
$64x^2 - y^4 + 64z^2 = 0$.

16.



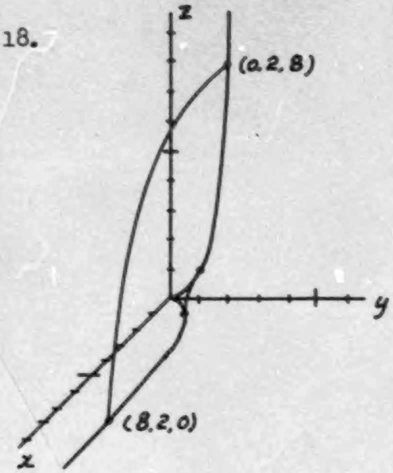
$9x^2 + 9y^2 - z^2 = 36$.

17.



$(x^2 + y^2)^3 - z^2 = 0$.

18.



$$x^2 - y^6 + z^2 = 0 .$$

19. Since this is a surface of revolution about the y-axis, any section parallel to the xz-plane will be a circle of radius k with equation $x^2 + z^2 = k^2$. The number k is the ordinate, z , of any point on the curve $f(y,z) = 0$ in the yz-plane; hence, since $k = \sqrt{x^2 + z^2}$, the equation of the surface is $f(y , \sqrt{x^2 + z^2}) = 0$.

372 Projecting cylinders, although time consuming to draw, can be very helpful in locating a space intersection. Look, for example, at Number 48(a) of the Review Exercises. We have the intersection of a spheroid and a hyperboloid; this is extremely difficult to visualize. But when we employ projecting cylinders, we see that the curve lies in a pair of planes through the y-axis and that its projection on the horizontal xy-plane is a circle. This is much easier to visualize. By the way, even if a student should use a different pair of projecting cylinders than we have used, he will obtain the same intersection.

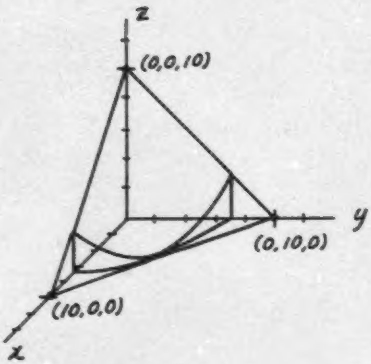
Exercises 9-7

1. (a)	$x^2 + z^2 = 12$, $y = -2$.	A circle; radius , $\sqrt{12}$; center on y-axis; parallel to and 2 units left of the xz-plane.
(b)	$y^2 + z^2 = -5$ $x = 3$.	No intersection. (This first equation represents an imaginary cylinder.)
(c)	$x^2 + y^2 = 4$, $z = 0$.	A circle; radius, 2 ; center at origin; in the xy-plane.
(d)	$x^2 + y^2 = 4$, $z = 0$.	Same locus as part (c) .
(e)	$x^2 + z^2 = 5$, $y = 5$.	A circle; radius, $\sqrt{5}$; center on y-axis; parallel to and 5 units right of the xz-plane.
(f)	$x^2 = 25$, $z = 0$.	A pair of lines; parallel to the y-axis; 5 units on opposite sides of the y-axis in the xy-plane.
(g)	$x^2 = 25$, $x - y = 0$.	A pair of lines; parallel to the z-axis; 5 units on opposite sides of the z-axis in the plane which bisects the first octant.
(h)	$x^2 + 8y^2 = 16$, $z = 1$.	An ellipse; center on z-axis; parallel to and 1 unit above the xy-plane.
(i)	$3y^2 - 4z^2 = 12$, $x = 0$.	A hyperbola; center at the origin; in the yz-plane.
(j)	$2y^2 + 8z^2 = 8$, $x = 0$.	An ellipse; center at the origin; in the yz-plane.
(k)	$x^2 + 8z^2 = 0$, $y = 2$.	The point (0,2,0) . (This first equation represents a degenerate elliptical cylinder--the y-axis.)

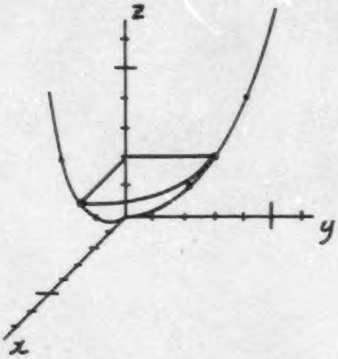
(ℓ) $x^2 + y^2 = 2$,
$z = 1$.

A circle; radius, $\sqrt{2}$; center on the z-axis; parallel to and one unit above the xy-plane. (Hint: subtract the second equation from the first and substitute for z .)

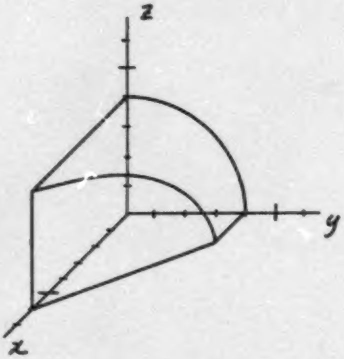
2. (a)


z
(0,0,10)
y
(0,10,0)
(10,0,0)
x

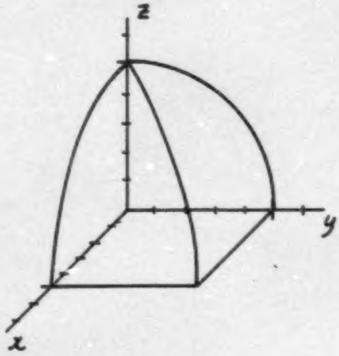
(c)



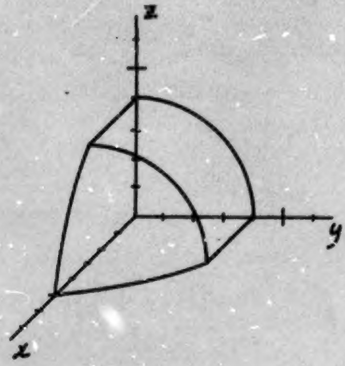
(b)



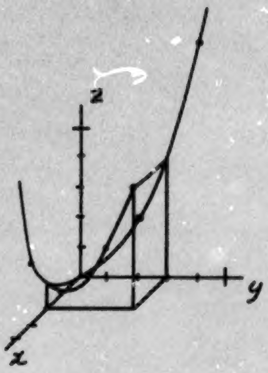
(d)



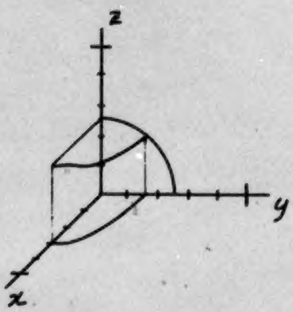
(e)



(f)



3.

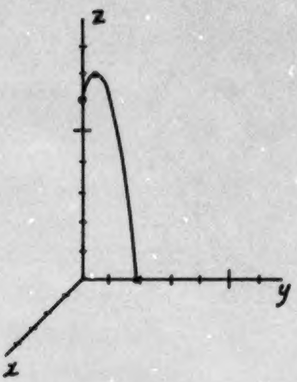


$y^2 + z^2 = 6$;

$-x^2 + 3z^2 = 9$;

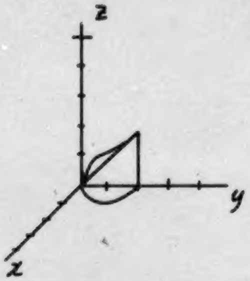
$x^2 + 3y^2 = 9$.

4.



The point is 14 units above the xy-plane. By eliminating x , the equation of the projecting cylinder with elements parallel to the x-axis is $z - 14 = -2(y - 1)^2$. The yz-trace of this parabolic cylinder is a parabola, which shows the projection of the highest point of the space curve, $(\frac{1}{2}, 1, 14)$. Interested students may wish to find this point by observing the projection in the xz-plane.

5.



$z^2 = x^2 + y^2$ represents one projecting cylinder, and eliminating x from the other equation gives $z^2 - 2y = 0$, which represents a projecting (parabolic) cylinder with elements parallel to the x-axis. The resulting space curve together with the xy- and yz-traces completes the outline of the figure.

377 Since cylindrical and spherical coordinates make use of polar forms, the remarks made previously about the ambiguity of this type of representation apply here as well.

There is nothing (except custom) to prevent us from applying the polar designations to one of the other coordinate planes. Thus, in Figure 9-23, we might have designated the point as $P = (r,y,\theta)$ or $P = (x,r,\theta)$. We chose the form which is in common use.

Exercises 9-8

1. $r = \rho \sin \phi$, $\theta = \theta$, $z = \rho \cos \phi$. or $\rho^2 = r^2 + z^2$, $\theta = \theta$, $\tan \phi = \frac{r}{z}$

$$\begin{aligned} r &= \rho \sin \phi , & & & \rho^2 &= r^2 + z^2 , \\ \theta &= \theta , & &\text{or} & \theta &= \theta \\ z &= \rho \cos \phi . & & & \tan \phi &= \frac{r}{z} \end{aligned}$$

These may be obtained by equating the cylindrical and spherical forms in terms of rectangular coordinates. For example,

$$r \cos \theta = x = \rho \sin \phi \cos \theta ,$$

$$r = \rho \sin \phi .$$

$$\tan \theta = \frac{y}{x} = \frac{\rho \sin \phi \sin \theta}{\rho \sin \phi \cos \theta} ,$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta} , \text{ (an identity); therefore,}$$

$$\theta = \theta .$$

2. (a) $(\sqrt{2}, \sqrt{6}, 2\sqrt{2})$; $(2\sqrt{2}, \frac{\pi}{3}, 2\sqrt{2})$.

   (b) $(\frac{3}{2}\sqrt{3}, 0, \frac{3}{2})$; $(\frac{3}{2}\sqrt{3}, 0, \frac{3}{2})$.

   (c) $(0,2,0)$; $(2, \frac{\pi}{2}, 0)$.

   (d) $(0.239, 3.354, 2.160)$; $(3.364, \frac{3}{2}, 2.160)$.

3. (a) $(\sqrt{3}, 1, 3)$; $(\sqrt{13}, \frac{\pi}{6}, .59)$.

   (b) $(0,5,0)$; $(5, \frac{\pi}{2}, \frac{\pi}{2})$.

   (c) $(0,0,8)$; $(8, \frac{\pi}{4}, 0)$.

   (d) $(2.160, 3.364, 2)$; $(2\sqrt{5}, 1, 1.11)$.

4. $\rho^2 = x^2 + y^2 + z^2$,

$\tan \theta = \frac{y}{x}$,

$\tan \phi = \frac{\sqrt{x^2 + y^2}}{z}$.

(a) $(\sqrt{13}, .98, 0)$; $(\sqrt{13}, .98, \frac{\pi}{2})$.

(b) $(6, \frac{\pi}{2}, 3)$; $(3\sqrt{5}, \frac{\pi}{2}, 1.11)$.

(c) $(4, \frac{\pi}{6}, 4)$; $(4\sqrt{2}, \frac{\pi}{6}, \frac{\pi}{4})$.

(d) $(\sqrt{17}, .24, 2)$; $(\sqrt{21}, .24, 1.34)$.

5. (a) $r^2 = 25$, or simply $r = 5$; $\rho^2 \sin^2 \phi = 25$, or $\rho \sin \phi = 5$.

(b) $z = 4 \tan \theta$; $\rho \cos \phi \cot \theta = 4$.

(c) $r = 8 \cos \theta$; $\rho \sin \phi = 8 \cos \theta$.

(d) $r^2 = 3z$; $\rho \sin \phi \tan \phi = 3$.

6. (a) $x^2 + y^2 + z^2 = 36$.

(b) $x^2 + y^2 = 36$.

(c) $x^2 + y^2 - (z - 6)^2 = 0$.

(d) $x^2 + y^2 + z^2 = 9$.

7. (a) A cylinder of radius 3 whose axis is the z-axis.

(b) A plane containing the z-axis and bisecting the first octant.

(c) A sphere of radius 2 with center at the origin.

(d) A circular cone whose vertex is at the origin and whose axis is the z-axis.

(e) A plane parallel to and 7 units above the xy-plane.

(f) A plane containing the y-axis and bisecting the first octant.

(g) A circular cone whose vertex is at the origin and whose axis is the z-axis.

(h) A plane parallel to and 2 units in front of the yz-plane.

8. (a) Let the center of the sphere be the origin and the axis of the cylinder have rectangular equations $x = 2$, $y = 0$; then the bounding surfaces are

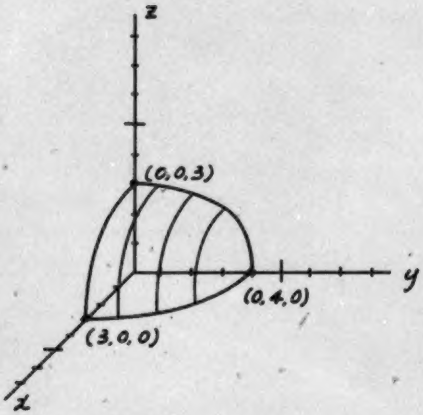
$$x^2 + y^2 + z^2 = 16 ,$$
$$(x - 2)^2 + y^2 = 4 .$$

(b) $r^2 + z^2 = 16$,
$r = 4 \cos \theta$.

(c) $\rho = 4$,
$\rho \sin \phi = 4 \cos \theta$.

Review Exercises

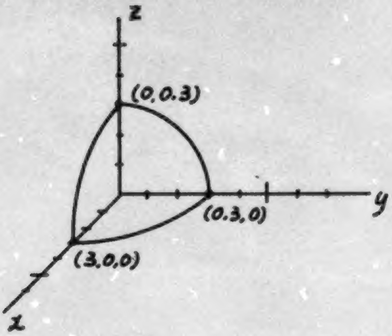
1.



A prolate spheroid.
Sections parallel to the xz-plane are circles. Sections parallel to the other coordinate planes are ellipses.

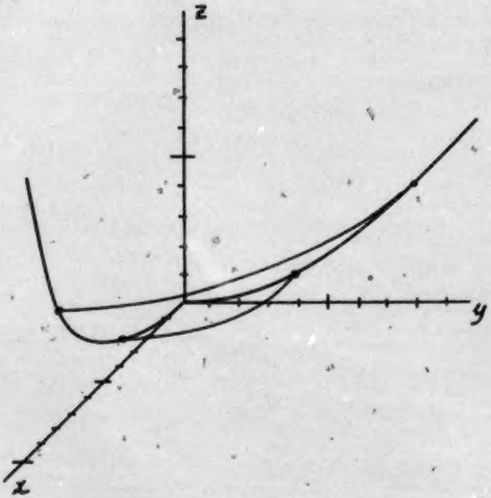
208

2.



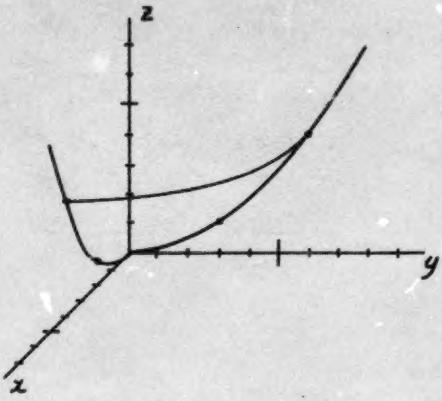
A sphere. Radius: 3 .
All sections are circles.

3.



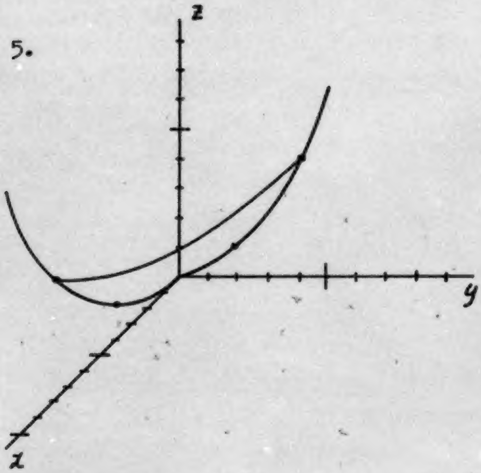
A paraboloid of revolution.
Sections parallel to the xy-plane are circles.
Sections parallel to the other coordinate planes are parabolas.

4.



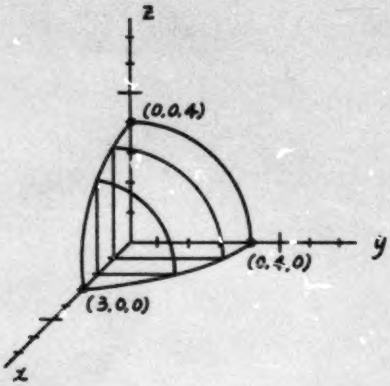
An elliptic paraboloid.
Sections parallel to the xy-plane are ellipses.
Sections parallel to the other coordinate planes are parabolas.

5.



An elliptic paraboloid.
Sections parallel to the xy-plane are ellipses.
Sections parallel to the other coordinate planes are parabolas.

6.

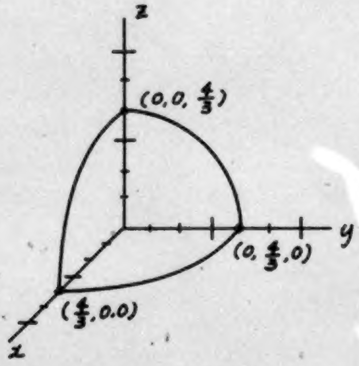

z
(0,0,4)
y
(0,4,0)
(3,0,0)
x

An oblate spheroid.

Sections parallel to the yz-plane are circles.

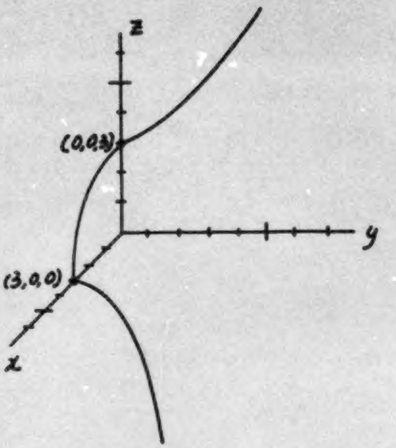
Sections parallel to the other coordinate planes are ellipses.

7.


z
$(0,0,\frac{4}{3})$
y
$(0,\frac{4}{3},0)$
$(\frac{4}{3},0,0)$
x

A sphere. Radius: $\frac{4}{3}$.

8.

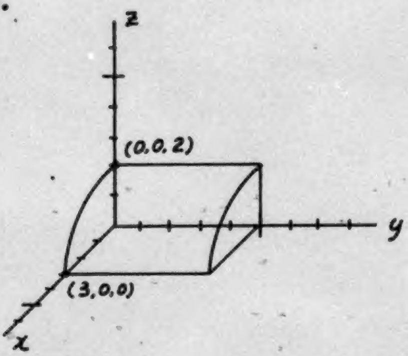


A hyperboloid of revolution (one sheet).

Sections parallel to the xz-plane are circles.

Sections parallel to the other coordinate planes are hyperbolas.

9.

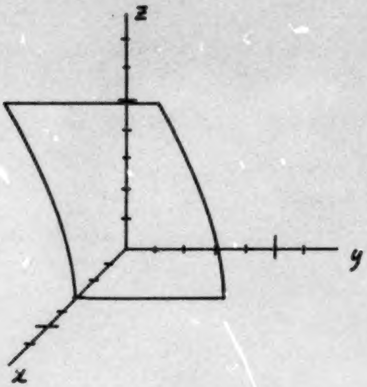


An elliptic cylinder.

Sections parallel to the xz-plane are ellipses.

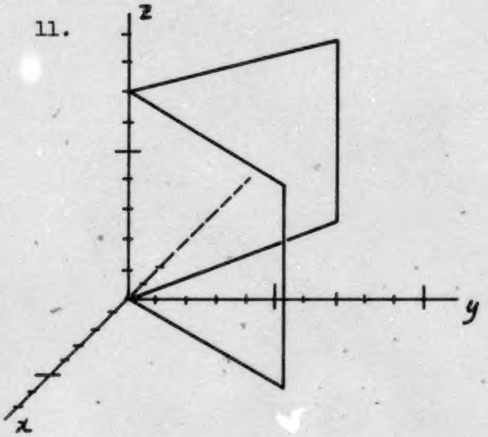
Sections parallel to the other coordinate planes are parallel lines.

10.



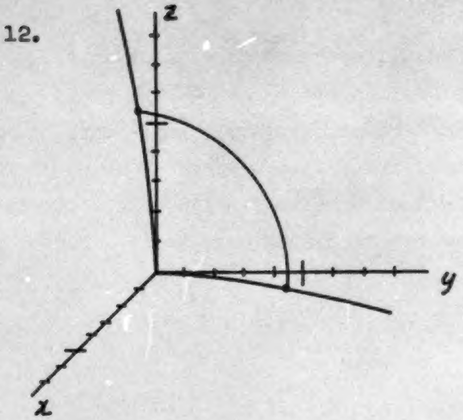
A hyperbolic cylinder (two parts). Sections parallel to the xz-plane are hyperbolas. Sections parallel to the other coordinate planes are parallel lines.

11.



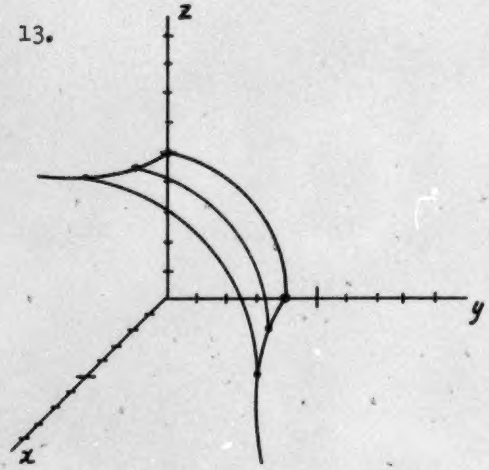
A pair of planes intersecting on the z-axis.

12.



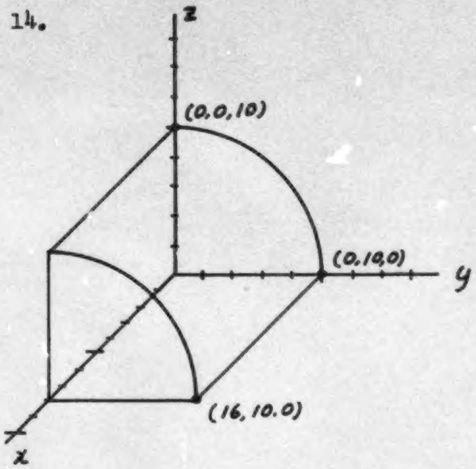
An elliptic paraboloid.
Sections parallel to the yz-plane are ellipses.
Sections parallel to the other coordinate planes are parabolas.

13.



An elliptic hyperboloid (one sheet).
Sections parallel to the yz-plane are ellipses.
Sections parallel to the other coordinate planes are hyperbolas.

14.

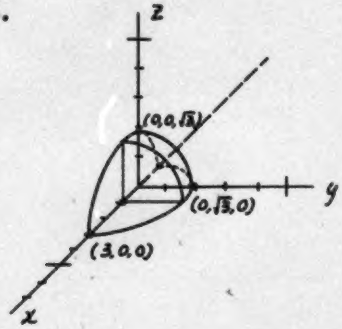

z
(0,0,10)
(0,10,0)
y
(16,10.0)
x

A circular cylinder; axis: x-axis; radius: 10 .

Sections parallel to the yz-plane are circles.

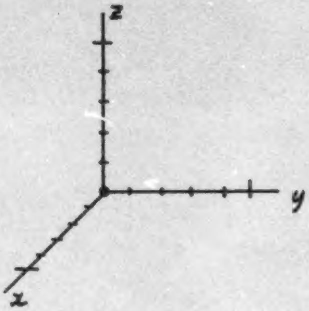
Sections parallel to the other coordinate planes are parallel lines.

15.


z
(0,0,√3)
(0,√3,0)
y
(3,0,0)
x

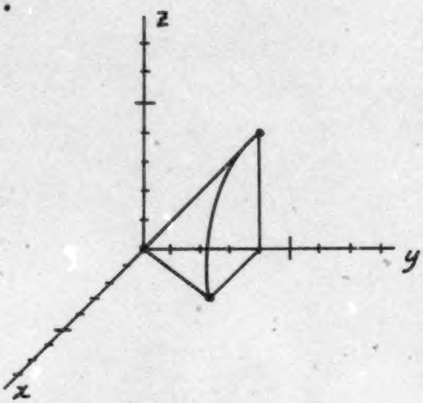
A sphere; center: (1,0,0) ; radius: 2 .

16.



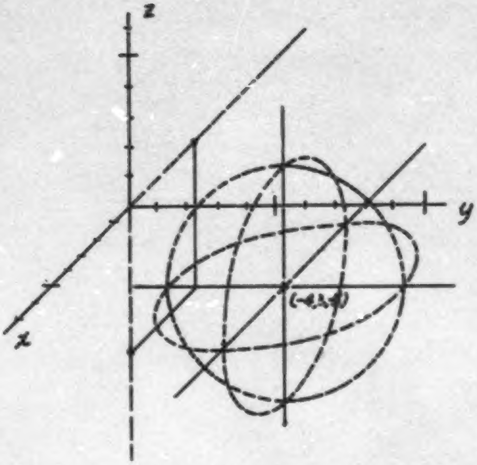
The point (0,0,0) .

17.



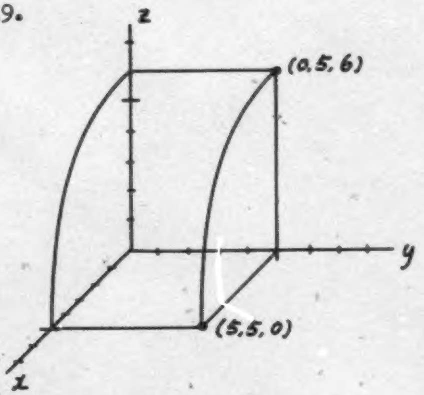
An elliptic cone.
Sections parallel to the xz-plane are ellipses.
Sections parallel to the other coordinate planes are hyperbolas.

18.



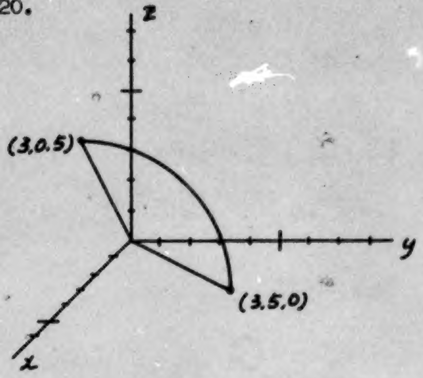
$(x + 4)^2 + (y - 3)^2 + (z + 5)^2 = 16$.
A sphere with radius 4 and center at $(-4,3,-5)$.

19.



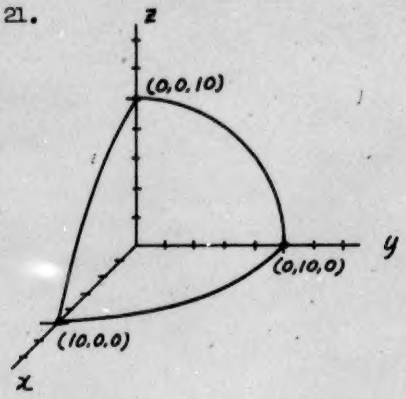
An elliptic cylinder.
Sections parallel to the xz-plane are ellipses.
Sections parallel to the other coordinate planes are parallel lines.

20.



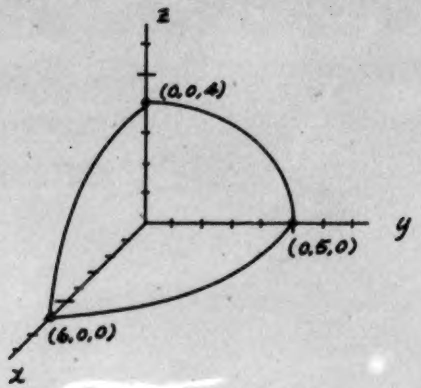
A circular cone.
Sections parallel to the yz-plane are circles.
Sections parallel to the other coordinate planes are hyperbolas.

21.


z
(0,0,10)
y
(0,10,0)
(10,0,0)
x

$x^2 + y^2 + z^2 = 100$.

22.

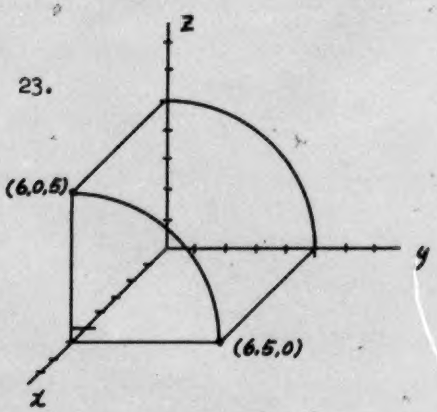

z
(0,0,4)
y
(0,5,0)
(6,0,0)
x

Assume that the ellipsoid has its center at the origin.

$$\frac{x^2}{36} + \frac{y^2}{25} + \frac{z^2}{16} = 1 ,$$

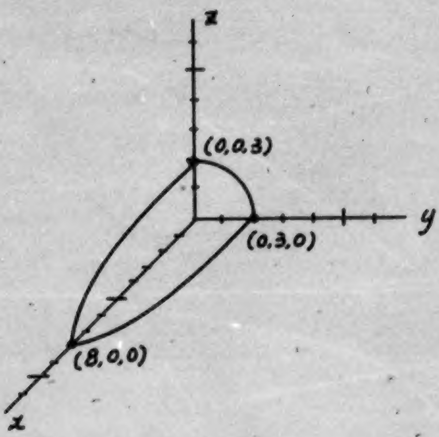
or $100x^2 + 144y^2 + 225z^2 = 3600$.

23.



$y^2 + z^2 = 25$.

24.

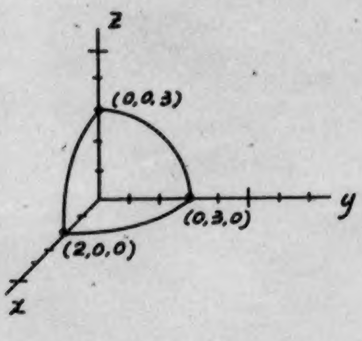

z
(0,0,3)
y
(0,3,0)
(8,0,0)
x

Assume the spheroid located as shown.

$$\frac{x^2}{64} + \frac{y^2}{4} + \frac{z^2}{4} = 1 \text{ , or}$$

$$x^2 + 16y^2 + 16z^2 = 64 .$$

25.

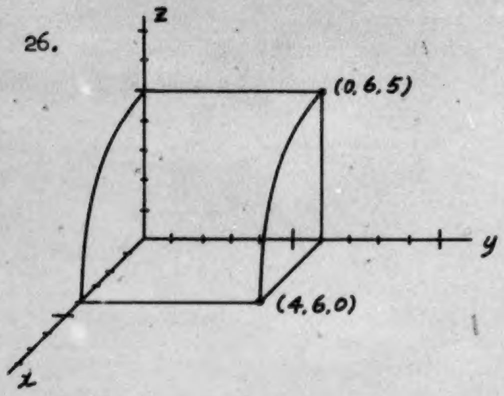

z
(0,0,3)
y
(0,3,0)
(2,0,0)
x

Assume the spheroid located as shown.

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{9} = 1 \text{ , or}$$

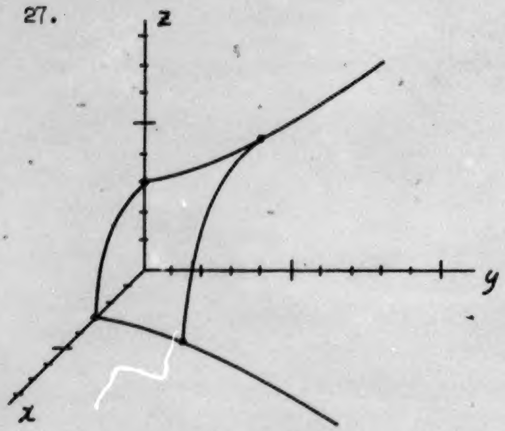
$$9x^2 + 4y^2 + 4z^2 = 36 .$$

26.



$25x^2 + 16z^2 = 400$.

27.

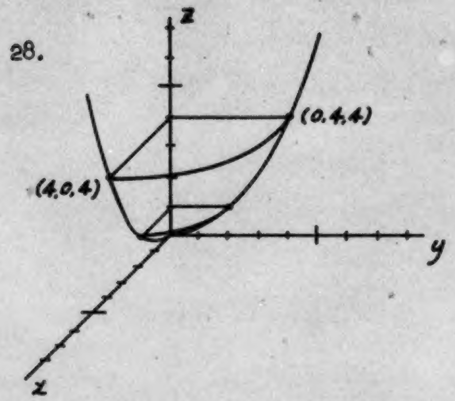


A hyperboloid of one sheet.

$\frac{x^2}{9} - \frac{y^2}{16} + \frac{z^2}{9} = 1$, or

$16x^2 - 9y^2 + 16z^2 = 144$.

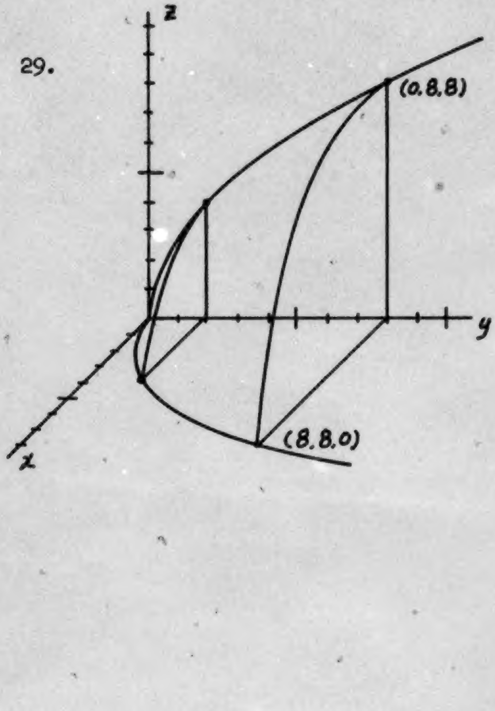
28.



A paraboloid.

$x^2 + y^2 = 4z$.

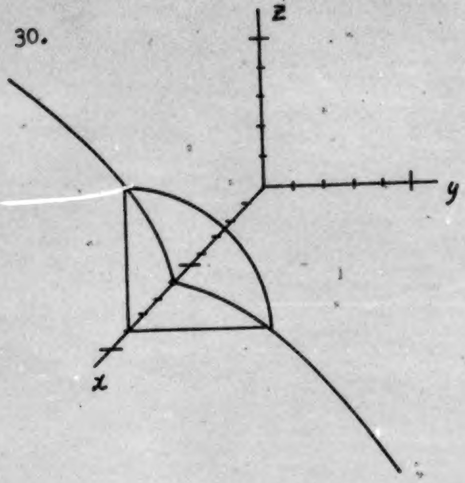
29.



A paraboloid.

$x^2 + z^2 = 8y$.

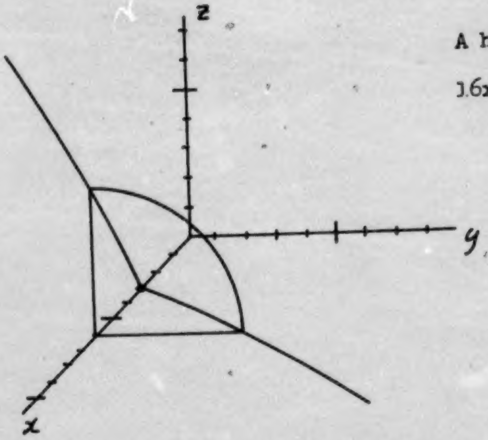
30.



A hyperboloid of two sheets.

$25x^2 - 36y^2 - 36z^2 = 900$.

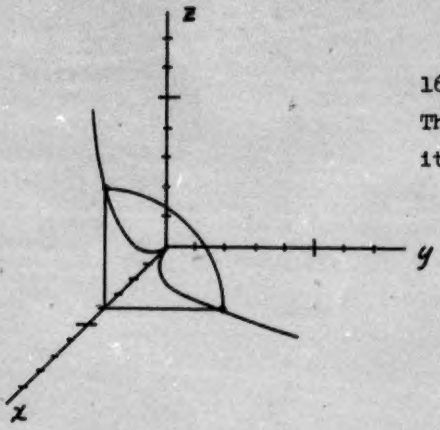
31.



A hyperboloid of two sheets.

$16x^2 - 9y^2 - 9z^2 = 144$.

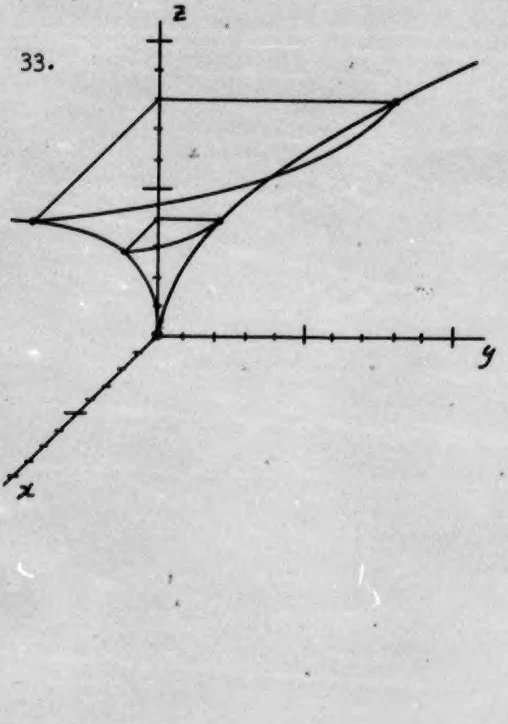
32.



$16y^2 + 16z^2 = x^4$.

This is not a quadric surface; it resembles Figure 9-19.

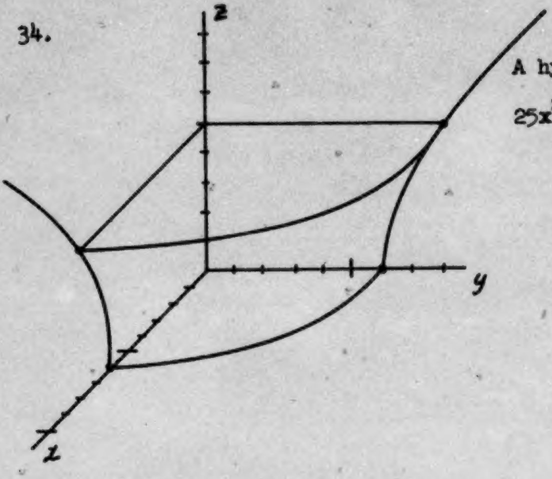
33.



$64x^2 + 64y^2 = z^4$.

Compare with Figure 9-19.

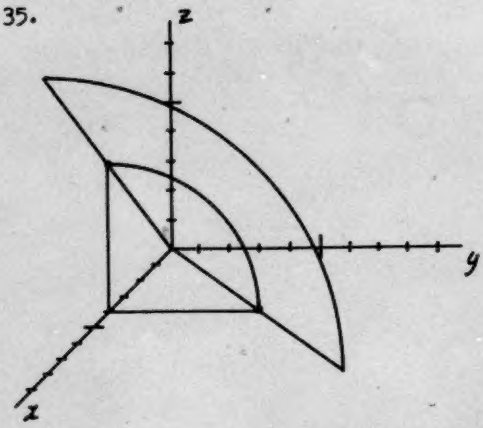
34.



A hyperboloid of one sheet

$25x^2 + 25y^2 - 36z^2 = 900$.

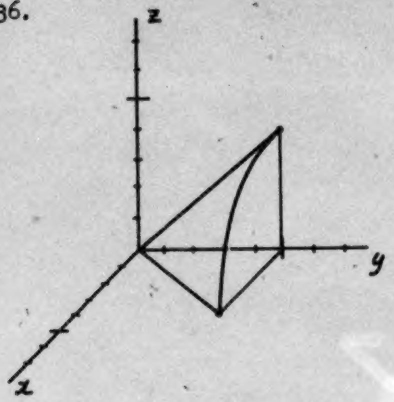
35.



A circular cone.

$-25x^2 + 16y^2 + 16z^2 = 0$

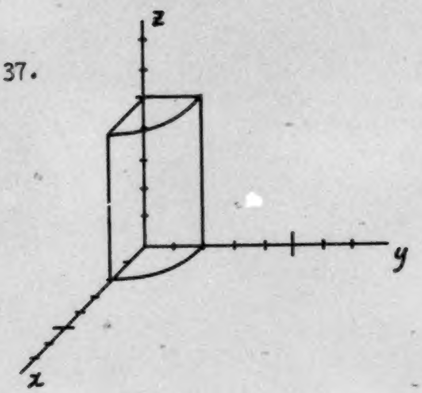
36.



A circular cone.

$25x^2 - 16y^2 + 25z^2 = 0$.

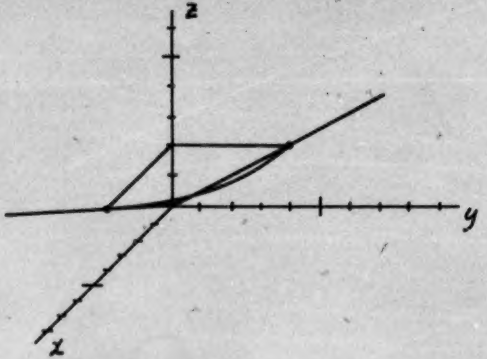
37.



A circular cylinder.

$x^2 + y^2 = 4$.

38.



A circular cone.

$x^2 + y^2 - 4z^2 = 0$.

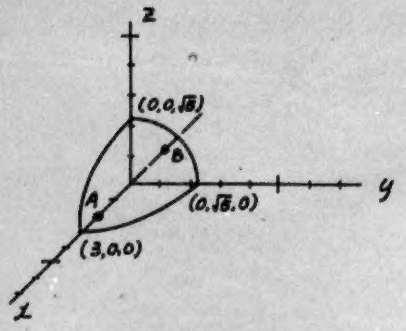
39. $x^2 + y^2 + z^2 - 6x + 4y - 2z - 86 = 0$.

40. $x^2 + z^2 = 25$.

41. $y^2 + z^2 = 12x - 36$.

42. (a) Ellipsoid.
    (b) A point.
    (c) No locus.
    (d) Elliptic hyperboloid of one sheet.
    (e) Elliptic cone.
    (f) Elliptic hyperboloid of two sheets.
    (g) Elliptic hyperboloid of two sheets.
    (h) Elliptic cone.
    (i) Elliptic hyperboloid of one sheet.

43.



Let the points be $A = (2,0,0)$ and $B = (-2,0,0)$. The equation is

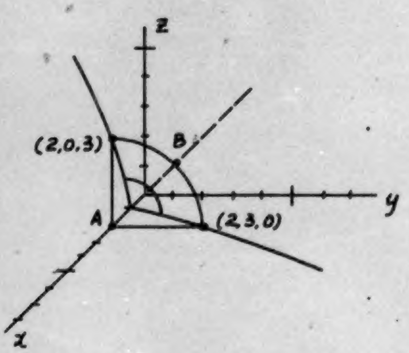
$$\sqrt{(x-2)^2+y^2+z^2} + \sqrt{(x+2)^2+y^2+z^2} = 6.$$

This simplifies to

$$5x^2 + 9y^2 + 9z^2 = 45 ,$$

an equation of a prolate spheroid.

44.



The equation is

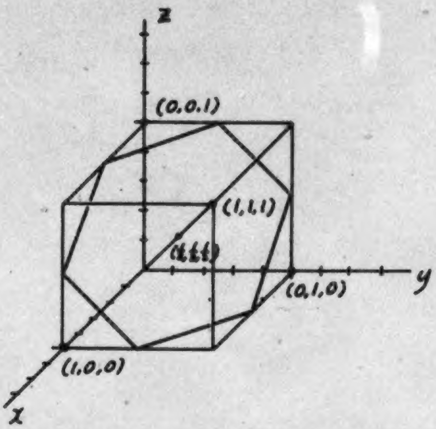
$$\sqrt{(x-2)^2+y^2+z^2} - \sqrt{(x+2)^2+y^2+z^2} = 2,$$

or $3x^2 - y^2 - z^2 = 3$,

an equation of a hyperboloid of two sheets.

45. Since the blades of the sharpener generate a circular cone and the sides of the pencil are portions of planes parallel to the axis of the cone, the intersection will be, under ideal conditions, portions of six congruent hyperbolas.

46.

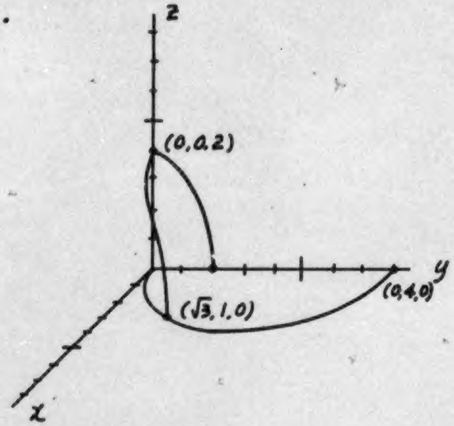


Since the plane contains the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, the equation of the plane in normal form is

$$\frac{1}{2}x + \frac{1}{2}y + \frac{1}{2}z = \frac{3}{4} \text{, or}$$

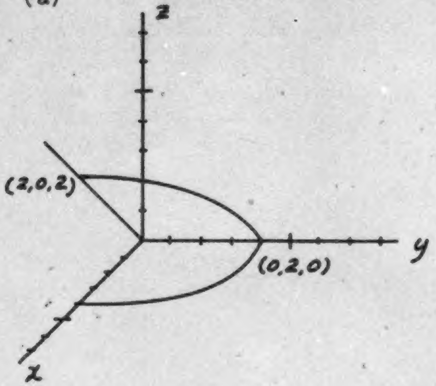
$x + y + z = \frac{3}{2}$. The intersection with the cube is a regular hexagon with sides of length $\frac{1}{2}\sqrt{2}$.

47.



The second equation represents a circular cylinder whose xy-trace is shown. Subtracting the second equation from the first gives $z^2 + 4y = 4$; this equation represents a parabolic cylinder whose yz-trace is shown. The space curve is the intersection of the sphere and cylinder.

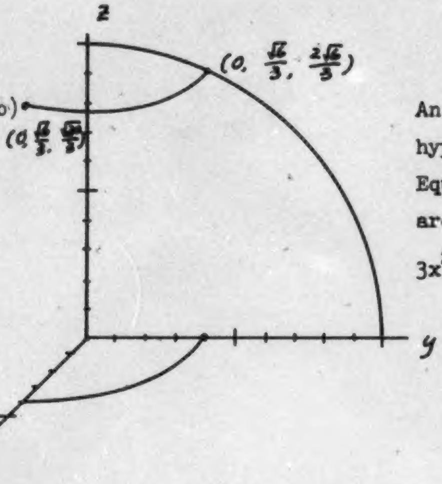
48. (a)



An oblate spheroid and an elliptic hyperboloid of one sheet. Two projecting cylinders have equations

$x^2 - z^2 = 0$, $x^2 + y^2 = 4$.

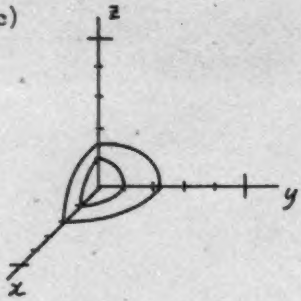
(b)


z
$(0, \frac{\sqrt{6}}{3}, \frac{2\sqrt{6}}{3})$
y
x

An oblate spheroid and an elliptic hyperboloid of two sheets. Equations of two projecting cylinders are

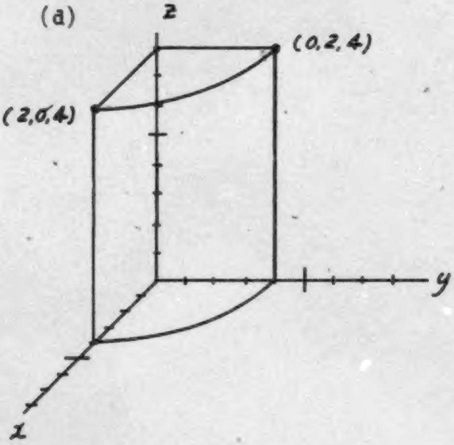
$3x^2 + 3y^2 = 2$, $3y^2 + 3z^2 = 10$.

(c)



A sphere and an oblate spheroid. These surfaces do not intersect. The sphere lies entirely inside the spheroid.

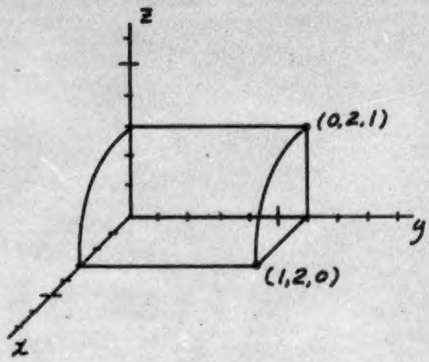
(d)



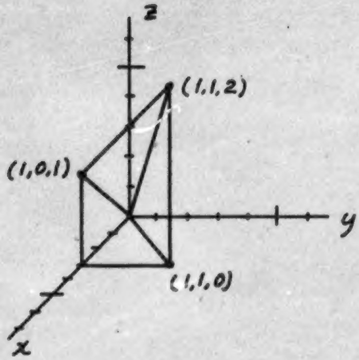
A paraboloid of revolution and a circular cylinder. Equations of two projecting cylinders are

$x^2 + y^2 = 4$, $z = 4$.

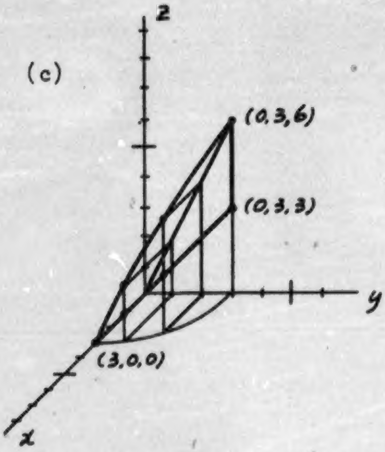
49. (a)


z
(0,2,1)
y
(1,2,0)
x

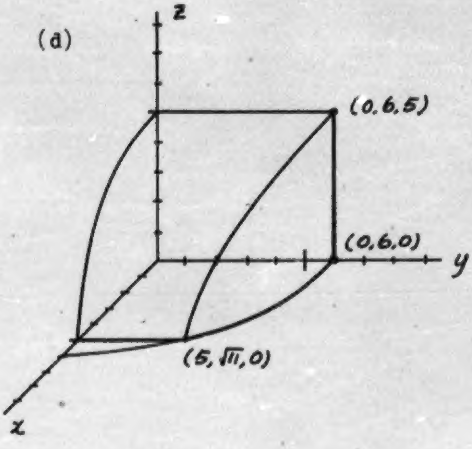
(b)


z
(1,1,2)
(1,0,1)
y
(1,1,0)
x

(c)


z
(0,3,6)
(0,3,3)
y
(3,0,0)
x

(d)


z
(0,6,5)
(0,6,0)
y
(5,$\sqrt{11}$,0)
x

50. (a) $z = 5$; $\rho \cos \phi = 5$.

(b) $r = 4 \cos \theta$; $\rho \sin \phi = 4 \cos \theta$.

(c) $x^2 + y^2 = 49$; $\rho \sin \phi = 7$.

(d) $r^2 + z^2 = 25$; $\rho = 5$.

(e) $x^2 + y^2 + z^2 = 9$; $\rho = 3$.

(f) $z = 6$; $z = 6$.

(g) $r^2(\cos^2 \theta - \sin^2 \theta) = 16$; $\rho^2 \sin^2 \phi (\cos^2 \theta - \sin^2 \theta) = 16$.

(h) $x^2 + y^2 = 2x$; $\rho \sin \phi = 2 \cos \theta$.

(i) $x^2 + y^2 = 2z$; $r^2 = 2z$.

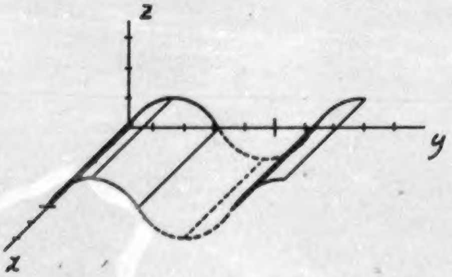
(j) $x^2 + y^2 = 9$; $r = 3$.

(k) $r = 8$; $\rho \sin \phi = 8$.

(ℓ) $x = yz$; $z = \cot \theta$.

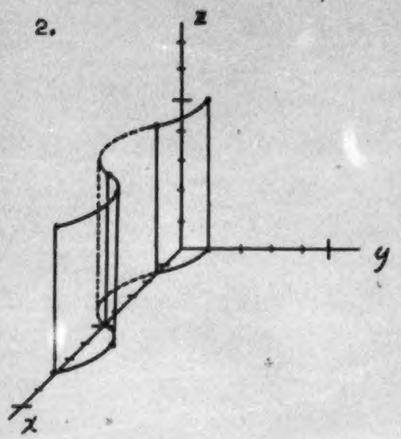
Challenge Problems

1.



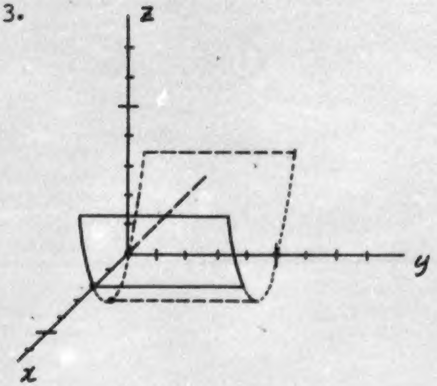
A cylinder with elements parallel to the x-axis and whose directrix is a sine curve.

2.



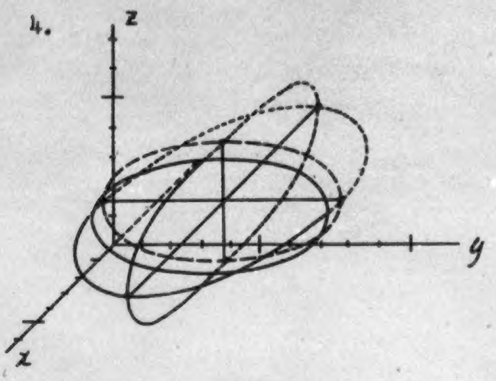
A cylinder with elements parallel to the z-axis and whose directrix is a cosine curve.

3.



A parabolic cylinder with elements parallel to the y-axis .

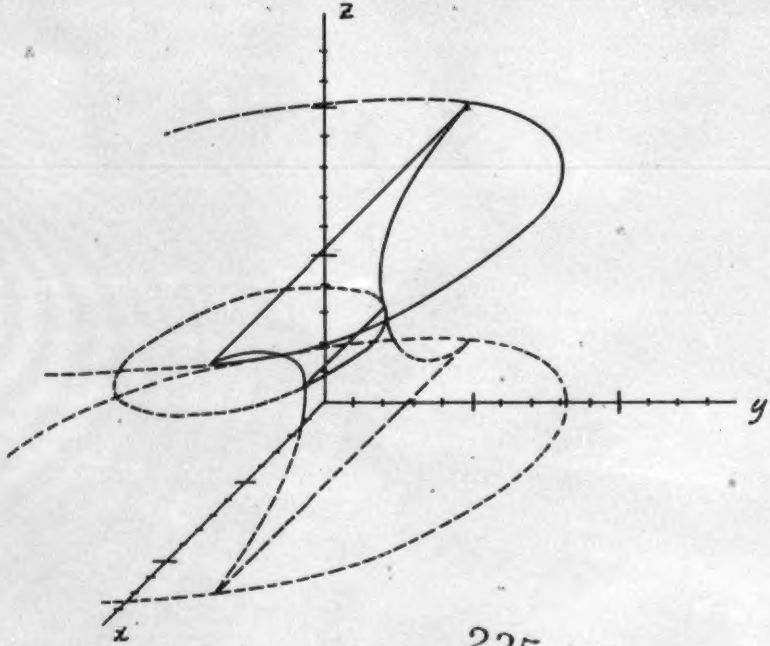
4.



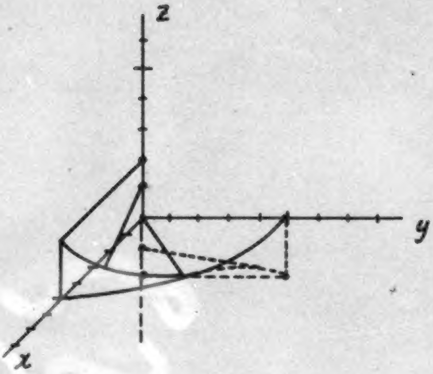
An ellipsoid with center at $(-1,3,1)$. The axis parallel to the x-axis has length 12 ; the axis parallel to the y-axis has length 8 ; the axis parallel to the z-axis has length 4 .

5.

A hyperboloid of revolution of one sheet with center at $(-1,-3,1)$.



6.



This curious surface is a type of ruled surface. It is entirely contained between the planes $z = 1$ and $z = -1$. It can be visualized as being generated by a line (minus the point on the z-axis) which is always perpendicular to the z-axis. Starting in the plane $z = 1$ and parallel to the x-axis, the line rotates as it drops until it is parallel to the y-axis in the plane $z = -1$. The line continues to rotate as it rises to its original position, thus completing the surface.

Teachers' Commentary

Chapter 10

GEOMETRIC TRANSFORMATION

10-1. Why Study Geometric Transformations?

Most of this chapter is an extension of Chapters 5-7 on curve tracing and conics. Although the principles presented in Sections 10-2 and 10-5 are applicable to all curves, we emphasized the straight line and the conics because of their importance and because the students are more familiar with them.

The treatment of geometric transformations in this text differs from most other texts in that we look upon geometric transformations from two points of view. We first move the axes, keeping the figure fixed; then we move the figure, keeping the axes fixed. We feel that the student should become acquainted with both types. The point transformation has much wider application than the transformation of axes and Section 6 was included to show its possibilities.

Groups of transformations are discussed in a supplementary chapter. You will recall that in 1872, Professor Felix Klein (1849-1925) presented his famous "Erlanger Program" in which he classified all geometrics on the basis of those properties invariant under groups of transformations. Mention is made in this chapter of the set of rigid motions which characterize Euclidean Geometry without designating them as a group. You will find a good treatment of this subject in Courant and Robbins' book entitled What is Mathematics?

10-2. Translations.

Sufficient motivation may be provided for this section by requiring the students to graph each part of the equations listed in the first section.

You will note that there are two forms given for the equations of translation. The form $\begin{cases} x' = x + h \\ y' = y + k \end{cases}$ is more useful for the translation of axes because more applications are similar to those presented in Examples 3 and 4.

NOTE: Almost all solutions for this chapter are presented without graphs since they are so familiar to you.

Exercises 10-2

1. $\begin{cases} x' = x + 3 \\ y' = y - 4 \end{cases}$,

   $O' = (-3,4)$

2. The equations of translation are $\begin{cases} x = x' + 3 \\ y = y' - 2 \end{cases}$.

   The new equation is:

   $$2(x' + 3)^2 - (y' - 2)^2 - 12(x' + 3) - 4(y' - 2) + 12 = 0 ,$$

   which simplifies to $2x'^2 - y'^2 = 2$.

3. (a) $(x' - 4)^2 + (y' - 6)^2 = r^2$. This is a circle with the same radius, and center at $(4,6)$.

   (b) $\dfrac{(x' - 4)^2}{a^2} - \dfrac{(y' - 6)^2}{b^2} = 1$.

   This is a congruent hyperbola with its center at $(4,6)$ and with its axes parallel to the x- and y-axes.

   Neither of the curves undergo a change. They merely have a new equation relative to the new axes.

4. The equations of translation are $\begin{cases} x' = x + 4 \\ y' = y + 2 \end{cases}$. The new coordinates of the vertices of the triangle are $A = (5,2)$, $B = (9,0)$, and $C = (7,6)$ with reference to the new origin. Two suggested methods are:

   (a) Application of the Pythagorean Theorem.

   (b) Proof that the product of the slopes of $\overline{AB}$ and $\overline{AC}$ equals -1 .

5. By completing the square, the equation of the hyperbola becomes

$$(1) \qquad (x + 5)^2 - (y - 2)^2 = 16 .$$

Substituting x' for $(x + 5)$ and y' for $(y - 2)$ into (1), we have

$$(2) \qquad x'^2 - y'^2 = 16 .$$

Equation (2) represents the same hyperbola with reference to the new axes with origin $O' = (-5,2)$.

To graph, translate the origin to O'. Draw the x'- and y'-axes through O'. Then draw the graph of Equation (2) with respect to the x'- and y'-axes.

6. The students may select any three points. We choose $A = (5,0)$, $B = (3,4)$, and $C = (0,5)$. After translation, the coordinates of $A = (4,1)$, $B = (2,5)$, and $C = (-1,6)$ with respect to the new origin. The transformed equation is $(x' + 1)^2 + (y' - 1)^2 = 25$. The new coordinates satisfy this equation.

7. After the first translation, L has the equation $3x' - 2y' + 11 = 0$ with respect to the x'- and y'-axes. After the second translation, L has the equation $3x'' - 2y'' + 13 = 0$ with respect to the x''- and y''-axes. The transformation $\begin{cases} x = x'' + 7 \\ y = y'' + 7 \end{cases}$ would have the same effect as the two successive ones.

There would be no difference in the final result if these translations were commuted. (This is true of all translations.)

8. (a) Completing the square, the equation becomes $(y - 3)^2 = 12(x + 1)$. The equations of translation are therefore:

$$\begin{cases} x' = x + 1 \\ y' = y - 3 \end{cases}$$

The parabola now has the equation $y'^2 = 12x'$ with respect to the new origin at $O' = (-1,3)$. To graph, draw the x'- and y'-axes through O' and sketch the new equation with respect to those axes.

The solution of the other parts is similar to part (a). The new equations and origin are:

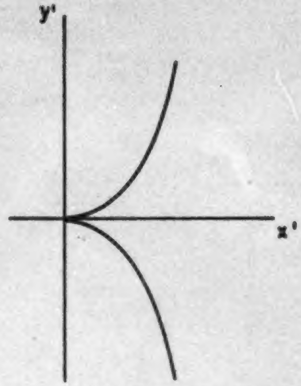
(b) $\frac{x'^2}{4} + \frac{y'^2}{3} = 1$; $O' = (1,-1)$.

(c) $x'^2 = \frac{3}{2}y'$; $O' = (-\frac{3}{2}, \frac{5}{2})$.

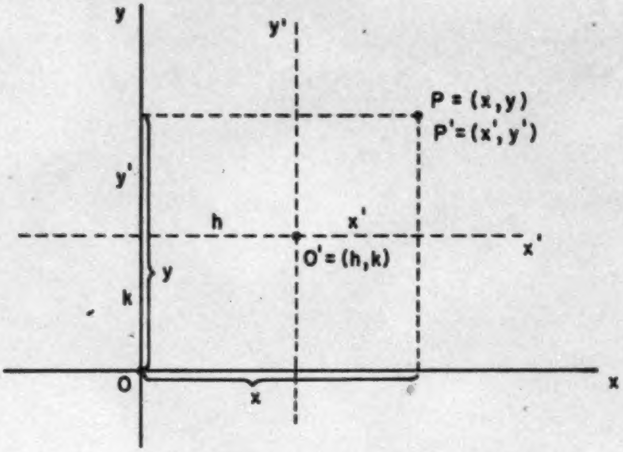
(d) $x'y' = 12$; $O' = (-3,4)$.

(e) $y'^2 = x'^3$; $O' = (-2,-2)$.

The graph of (e) looks like the figure to the right. It is called a semi-cubic parabola. There is no asymptote.



9.



From the figure, $\begin{cases} x = x' + h \\ y = y' + k \end{cases}$ or $\begin{cases} x' = x - h \\ y' = y - k \end{cases}$.

10-3. Rotation of Axes. Rectangular Coordinates.

Motivation for this section, as for Section 10-2, could be provided by asking the students to graph the pair of equations

$$x^2 + 2\sqrt{3}\,xy - y^2 = 8 \quad \text{and} \quad x'^2 - y'^2 = 4 .$$

Then point out to them or have them discover that the graphs are identical except for position.

You will note that we present two forms for the equations of rotation. The form chosen for use in solving a given problem depends upon the nature and form of the problem. The four examples presented in the text should clarify this point.

Your better students should be encouraged to study the Supplement to Chapter 7 where the topic is discussed in detail. Among other things, the student will learn how to determine the angle of rotation in order to arrive at a new set of axes and an equation containing no xy-term.

402 The "digression" on this page, which discusses the merits of one form of an equation over another for the same curve when both are simple, has an ulterior motive. That purpose is to indicate several scientific areas in which the equilateral hyperbola, in the form $xy = k$, is studied. As a rule, students are more acquainted with the other conics.

403 We recommend that the details of this rotation be carried out in class by the instructor. The students may then carry out the details of rotating the axes through an angle of measure α, and arrive at the equation of the circle

$$x'^2 + y'^2 + Dx' + Ey' + F = 0$$

with respect to the new $x'y'$-axes. No $x'y'$-term should appear. A complete discussion of the general equation of the second degree is found in the Supplementary Chapter for Chapter 7.

Because of the nature of polar coordinates, the rotation of the polar axis leads to a very simple result. Once again we have restricted ourselves to the conics. If time permits, you may like to discuss the rose curves, lemniscates, spirals, and other curves.

Exercises 10-3

1. Since $\alpha = 150^\circ$, $\sin\alpha = \frac{1}{2}$ and $\cos\alpha = -\frac{\sqrt{3}}{2}$. The equations of rotation are $\begin{cases} x' = \frac{1}{2}(-\sqrt{3}x + y) \\ y' = \frac{1}{2}(-x - \sqrt{3}y) \end{cases}$. The new coordinates of the vertices of the triangle are $A = (-\frac{\sqrt{3}}{2}, -\frac{1}{2})$, $B = \left(\frac{-5\sqrt{3} - 2}{2}, \frac{-5 + 2\sqrt{3}}{2}\right)$, and $C = \left(\frac{-3\sqrt{3} + 4}{2}, \frac{-3 - 4\sqrt{3}}{2}\right)$. Using the original coordinates, we

have $d(A,B) = \sqrt{20}$, $d(A,C) = \sqrt{20}$. $\therefore$ Area of $\triangle ABC = 10$. Using the new coordinates, we have

$$d(A,B) = \sqrt{\left(\frac{-5\sqrt{3} - 2}{2} + \frac{\sqrt{3}}{2}\right)^2 + \left(\frac{-5 + 2\sqrt{3}}{2} + \frac{1}{2}\right)^2}$$

$$= \sqrt{(-2\sqrt{3} - 1)^2 + (-2 + \sqrt{3})^2} = \sqrt{20} .$$

Similarly $d(A,C) = \sqrt{20}$, and the area of $\triangle ABC = 10$.

2. Since $\alpha = -30^\circ$, $\sin \alpha = -\frac{1}{2}$ and $\cos \alpha = \frac{\sqrt{3}}{2}$. The equations of rotation are therefore:

$$\begin{cases} x = \frac{1}{2}(\sqrt{3}\, x' + y') \\ y = \frac{1}{2}(-x' + \sqrt{3}\, y') \end{cases} .$$

The equation of the line with respect to the new axes is:

$$\frac{3}{2}(\sqrt{3}\, x' + y') + 1(-x' + \sqrt{3}\, y') - 8 = 0 ,$$

which simplifies to $(3\sqrt{3} - 2)x' + (2\sqrt{3} + 3)y' - 16 = 0$.

The slope is $-\dfrac{3\sqrt{3} - 2}{2\sqrt{3} + 3} = \dfrac{13\sqrt{3} - 24}{3}$.

3. $$\begin{cases} x' \cos \alpha - y' \sin \alpha = x \\ x' \sin \alpha + y' \cos \alpha = y \end{cases}$$

$$\therefore \quad x' \cos^2 \alpha - y' \sin \alpha \cos \alpha = x \cos \alpha$$

$$x' \sin^2 \alpha + y' \sin \alpha \cos \alpha = y \sin \alpha$$

Adding corresponding members, we have

$$x'(\cos^2 \alpha + \sin^2 \alpha) = x \cos \alpha + y \sin \alpha$$

or $$x' = x \cos \alpha + y \sin \alpha .$$

Likewise, $$y' = -x \sin \alpha + y \cos \alpha .$$

4. Since $\alpha = 45^\circ$, the equations of rotation are

$$\begin{cases} x = \frac{1}{\sqrt{2}}(x' - y') \\ y = \frac{1}{\sqrt{2}}(x' + y') \end{cases} .$$

The new equation is $\dfrac{(x' - y')^2}{2} = \dfrac{x' + y'}{\sqrt{2}}$ which simplifies to

$x'^2 - 2x'y' + y'^2 - \sqrt{2}\, x' - \sqrt{2}\, y' = 0$.

5. The solution is similar to that of Exercise (4) . The answers are:

(a) $x'^2 + 5y'^2 = 6$.

(b) (Here $\sin\theta = \frac{\sqrt{5}}{5}$, $\cos\theta = \frac{2\sqrt{5}}{5}$) .

$25x'^2 + 13y'^2 = 25$.

(c) $y'^2 - x'^2 = 8$.

(d) $x'^2 = -4y'$

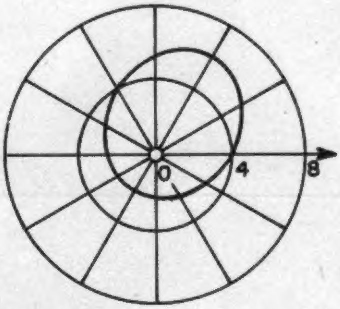
6. After rotation, we have

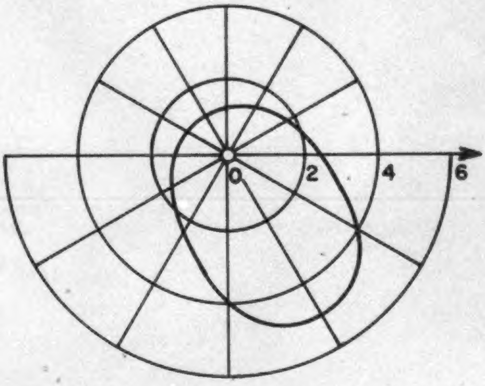
$$(x' \cos\alpha - y' \sin\alpha)^2 + (x' \sin\alpha + y' \cos\alpha)^2 = r^2 ,$$

or
$$x'^2 \cos^2\alpha - 2x'y' \sin\alpha\cos\alpha + y'^2 \sin^2\alpha +$$
$$x'^2 \sin^2\alpha + 2x'y' \sin\alpha\cos\alpha + y'^2 \cos^2\alpha = r^2 .$$

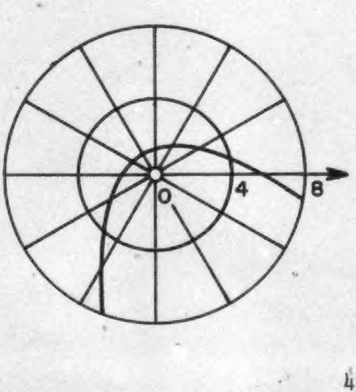
Thus, $x'^2 + y'^2 = r^2$.

7. (a) $r = r\,\frac{6}{2 - \cos(\theta - 60^\circ)}$ (b) $r = \frac{10}{5 + 3\cos(\theta - 120^\circ)}$





(c) $r = \frac{3}{1 + \sin(\theta - 30^\circ)}$



Challenge Problems

1. The proof is as follows: after rotation of axes, the new equation is

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0 \quad \text{where}$$

$$A' = A\cos^2\alpha + B\sin\alpha\cos\alpha + C\sin^2\alpha$$

$$B' = -2A\sin\alpha\cos\alpha + B\cos^2\alpha - B\sin^2\alpha + 2C\sin\alpha\cos\alpha$$

$$C' = A\sin^2\alpha - B\sin\alpha\cos\alpha + C\cos^2\alpha$$

$$D' = D\cos\alpha + E\sin\alpha$$

$$E' = -D\sin\alpha + E\cos\alpha$$

$$F' = F$$

When you perform the indicated operations and simplify, you find that

$$B'^2 - 4A'C' = B^2 - 4AC .$$

2. $$x = x'\cos\alpha - y'\sin\alpha = (x''\cos\theta - y''\sin\theta)\cos\alpha - (x''\sin\theta + y''\cos\theta)\sin\alpha$$

$$x''(\cos\theta\cos\alpha - \sin\theta\sin\alpha) - y''(\sin\theta\cos\alpha + \cos\theta\sin\alpha)$$

$$\therefore \quad x = x''\cos(\theta + \alpha) - y''\sin(\theta + \alpha) .$$

Likewise, $y = x''\sin(\theta + \alpha) + y''\cos(\theta + \alpha)$

10-4. Invariant Properties.

We have already touched upon the significance of the study of the geometric properties invariant under certain transformations. When the axes are rotated or translated, and the figure remains fixed, the question of invariant properties has meaning only with respect to observers using different points or lines of reference. When we study point transformations in the next two sections, the question of invariant properties has many more and varied aspects. When the points of a figure are moved, we are frequently not certain about the appearance of the image; it may or may not be congruent to the original figure.

You may wish to omit the discussion following Theorem 10-3. It is included to show a second approach to the problem and to lead to an interesting challenge exercise.

The exercises for this section were deliberately selected to point out properties other than distance and angle which remain invariant under the set of translations and rotations. We encourage you to discuss these other properties carefully on the basis of the exercises. The students should be encouraged to find more invariants than are indicated.

Exercises 10-4

1. (a) $3x + 2y - 8 = 0$

   (b) The equations of translation are

$$\begin{cases} x' = x + 4 \\ y' = y + 6 \end{cases} \quad \text{or} \quad \begin{cases} x = x' - 4 \\ y = y' - 6 \end{cases} .$$

   Thus $A = (6,7)$ and $B = (4,10)$ with respect to the new origin. Also, with respect to the new origin, the line has the equation $3(x' - 4) + 2(y' - 6) - 8 = 0$, which simplifies to $3x' + 2y' - 32 = 0$.

   (c) $d(A,B) = \sqrt{13}$ with respect to either set of axes.

2. The equations of rotation are

$$\begin{cases} x' = x \cos \alpha + y \sin \alpha = y \\ y' = -x \cos \alpha + y \cos \alpha = -x \end{cases}$$

   Thus $A = (1,-2)$ and $B = (4,0)$ with respect to the new axes. $d(A,B) = \sqrt{13}$ with respect to either set of axes.

3. The equations of translation are

$$\begin{cases} x' = x + 1 \\ y' = y + 1 \end{cases} \quad \text{or} \quad \begin{cases} x = x' - 1 \\ y = y' - 1 \end{cases} .$$

   (a) $A' = (1,-3)$, $B' = (3, -\frac{1}{3})$ and $C' = (4,1)$.

   L' has the equation $4(x' - 1) - 3(y' - 1) - 12 = 0$, which reduces to $4x' - 3y' - 13 = 0$.

   (b) B is between A and C since $d(A,B) + d(B,C) = d(A,C)$. B' is between A' and C' since $d(A',B') + d(B',C') = d(A',C')$.

   (c) Since $d(A',B') + d(B',C') = d(A',C')$, the points are collinear. Another way to prove collinearity is to show that the slopes of $\overline{A'B'}$, $\overline{B'C'}$, and $\overline{A'C'}$ are equal since B' is common to $\overline{A'B'}$ and $\overline{B'C'}$.

4. (a) The lines are concurrent since the point $(2,1)$ lies on all three lines.

   (b) The equations of translation are:

$$\begin{cases} x' = x - 3 \\ y' = y + 2 \end{cases} \quad \text{or} \quad \begin{cases} x = x' + 3 \\ y = y' - 2 \end{cases} .$$

The equations of the three lines with respect to the new axes are:

$$L_1' : 4x' - 3y' + 13 = 0$$

$$L_2' : x' - 2y' + 7 = 0$$

$$L_3' : 5x' - 3y' + 14 = 0 \ .$$

(c) The lines are concurrent since the point $(-1,3)$ lies on all three lines.

(d) Point $(2,1)$ maps into $(-1,3)$ under this translation.

(e) When the axes are rotated through an angle of 45°, the equations of rotation are

$$\begin{cases} x' = \frac{1}{\sqrt{2}}(x + y) \\ y' = \frac{1}{\sqrt{2}}(-x + y) \end{cases} \quad \text{or} \quad \begin{cases} x = \frac{1}{\sqrt{2}}(x' - y') \\ y = \frac{1}{\sqrt{2}}(x' + y') \ . \end{cases}$$

(1) The equations of the three lines with respect to the new axes are:

$$L_1' : x' - 7y' - 5\sqrt{2} = 0$$

$$L_2' : x' + 3y' = 0$$

$$L_3 : 2x' - 8y' - 7\sqrt{2} = 0$$

(2) The lines are concurrent since the point $(\frac{3}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ lies on all three lines.

(3) Point $(2,1)$ maps into $(\frac{3}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ under this rotation.

5. (a) $m_1 = -\frac{3}{2}$, $m_2 = 5$

$$\cos\theta = \frac{1 + m_1 m_2}{\sqrt{1 + m_1^2}\ \sqrt{1 + m_2^2}} = -\frac{\sqrt{2}}{2} \ .$$

$\therefore\ \theta = \frac{3\pi}{4}$ and its supplement is $\frac{\pi}{4}$.

(b) The equations of translation are

$$\begin{cases} x = x' + 2 \\ y = y' + 2 . \end{cases}$$

With respect to the new axes, the equations of the lines are:

$$L_1' : 3x' + 2y' + 2 = 0$$

$$L_2' : 5x' - y' - 1 = 0$$

Since $m_1' = -\frac{3}{2}$, $m_2' = 5$, $\theta' = \frac{\pi}{4}$(and $\frac{3\pi}{4}$).

Challenge Problem

The proof that the measure of angle is invariant under rotation may be presented as follows:

(1) Consider the angle between the lines

$$L_1 : a_1x + a_2y + a_3 = 0$$

$$L_2 : b_1x + b_2y + b_3 = 0 .$$

It will be convenient to use the formula for the angle between two lines in the form: $\tan \alpha = \dfrac{m_2 - m_1}{1 + m_1m_2}$ which is equivalent to the cosine form developed in this text. Thus:

$$\tan \alpha = \frac{\frac{-b_1}{b_2} + \frac{a_1}{a_2}}{1 + \frac{a_1b_1}{a_2b_2}} = \frac{a_1b_2 - a_2b_1}{a_1b_1 + a_2b_2}$$

(2) The equations of rotation are

$$x = x' \cos \theta - y' \sin \theta$$

$$y = x' \sin \theta + y' \cos \theta .$$

Substituting in L_1 and L_2, we have

$$L_1' : (a_1 \cos \theta + a_2 \sin \theta)x' + (a_2 \cos \theta - a_1 \sin \theta)y' + a_3 = 0.$$

$$L_2' : (b_1 \cos \theta + b_2 \sin \theta)x' + (b_2 \cos \theta - b_1 \sin \theta)y' + b_3 = 0.$$

The angle between L_1 and L_2 is given by

$$\tan \alpha' = \frac{(a_1\cos\theta + a_2\sin\theta)(b_2\cos\theta - b_1\sin\theta) - (a_2\cos\theta - a_1\sin\theta)(b_1\cos\theta + b_2\sin\theta)}{(a_1\cos\theta + a_2\sin\theta)(b_1\cos\theta + b\ \sin\theta) + (a_2\cos\theta - a_1\sin\theta)(b_2\cos\theta - b_1\sin\theta)}$$

(3) This complicated expression reduces to

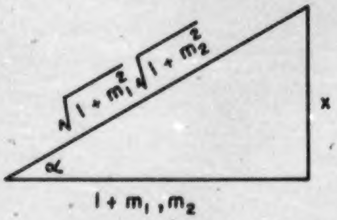
$$\tan \alpha' = \frac{a_1b_2 - a_2b_1}{a_1b_1 + a_2b_2} = \tan \alpha$$

after several applications of the identity $\sin^2 \alpha + \cos^2 \alpha = 1$.

Thus $\alpha = \alpha'$ for the principal values of $\tan \alpha$ and $\tan \alpha'$.

NOTE: Before offering this problem, you may wish to show the equivalence of the two formulas: $\cos \alpha = \dfrac{1 + m_1m_2}{\sqrt{1 + m_1^2}\ \sqrt{1 + m_2^2}}$ and $\tan \alpha = \dfrac{m_2 - m_1}{1 + m_2m_1}$.

The equivalence follows from the Pythagorean Theorem and the definitions of the trigonometric functions.



$$x^2 + (1 + m_1m_2)^2 = (1 + m_1^2)(1 + m_2^2) .$$

Thus $x = \pm (m_2 - m_1)$ and $\tan \alpha = \dfrac{\pm (m_2 - m_1)}{1 + m_2m_1}$. This formula gives us the angle and its supplement. We chose $+(m_2 - m_1)$ arbitrarily.

10-5. Point Transformations.

Most of the transformations studied by mathematicians are considered as point transformations or mappings. The reason for dicussing the transformation of axes first is that this type is most useful in reducing a complicated equation to a simpler form for sketching. Considerable care has been taken to distinguish between the two sets of transformations and to indicate that translations and rotations can be effected by either type.

The material included here on reflections relates so closely to the discussion of symmetry in Section 6-2 that a review of that topic may be appropriate before proceeding with this subject.

Euclidean geometry in characterized by the fact that the measures of both distance and angle are preserved under translation and rotation. In elementary geometry this statement is expressed in the "Postulate for Rigid Motion" which states that an object may be moved in space without changing its size and shape. We now see that all rigid motion can be performed by a series of no more than three reflections.

The SMSG Geometry, Appendix 8, has an excellent discussion on rigid motion.

The students may wonder why the third reflection is necessary in Example 3. Two reflections are sufficient if the sequence of points on the line is not considered. You might label several points on $\overline{A'B'}$, see where they fall on $\overline{CD}$, and then observe what happens after the second and third reflections.

An interesting result is obtained by subtracting the corresponding members of the equations of circles C and C' described on this page. The result is $8x + 12y = 0$ or $y = \frac{2}{3}x$. This is the equation of the common chord (or radical axis) shown.

A reflection is an example of what is called an "involutory transformation". A transformation is called an involutory transformation if it has the property that, if repeated once, it produces the identity transformation. This can be written analytically as follows: Let $(x,y) \longrightarrow (x'y') \longrightarrow (x'',y'')$. If $x'' = x$ and $y'' = y$, then the transformation is involutory.

Exercises 10-5

1. (a) $A = (1,2)$ maps into $A' = (1,-2)$ and $B = (3,-4)$ maps into $B' = (3,4)$ after reflection with respect to the x-axis.

$$d(A,B) = 2\sqrt{10} = d(A',B') .$$

(b) $A = (1,2)$ maps into $A' = (-1,2)$ and $B = (3,-4)$ maps into $B'(-3,-4)$ after reflection with respect to the y-axis.

$$d(A,B) = 2\sqrt{10} = d(A',B') .$$

(c) $A = (1,2)$ maps into $A' = (-1,-2)$ and $B = (3,-4)$ maps into $B' = (-3,4)$ after reflection with respect to the origin.

$$d(A,B) = 2\sqrt{10} = d(A',B') .$$

(d) $A = (1,2)$ maps into $A' = (11,2)$ and $B = (3,-4)$ maps into $B' = (9,-4)$ after reflection with respect to the line $x = 6$.

$$d(A,B) = 2\sqrt{10} = d(A',B') .$$

2. We choose the points $A = (2,0)$, $B = (4,0)$, $C = (7,0)$. Under this transformation, the images of these points are $A' = (4,0)$, $B' = (6,0)$, and $C' = (9,0)$. Two invariant properties are the measure of distance and the order of the points on the line. The three points on the line also remain collinear (a third invariant property).

3. Under the transformation $x' = 2x$, the images of the three points are $A' = (4,0)$, $B' = (8,0)$, and $C' = (14,0)$. Three invariant properties are: the origin remains fixed, i.e., $(0,0) \longrightarrow (0,0)$, the order of the three points on the line, and collinearity. (Note that distance is not an invariant property under this transformation.)

4. The angle between L_1 and L_2 has measure 45° . When both lines are rotated through an angle of measure $\frac{\pi}{4}$, the equation of L_1 becomes $y' = x'$ and the equation of L_2 becomes $x' = 0$. The angle between these lines also has measure 45° .

(NOTE: This problem can, of course, be solved by using the equations of rotation. Since the lines are rotated through an angle of measure $\frac{\pi}{4}$, the axes must be rotated through angle of measure $-\frac{\pi}{4}$ to achieve the same result.)

5. The images are:

(a) $y^2 = -x$

(b) $x^2 = -y$

(c) $xy = 6$

(d) $x^2 - y^2 = 1$

(e) $x^2 + y^2 + 2x + 4y + 4 = 0$

(f) $y = -x^3$

(g) $y = -\sin x$

(h) $y = -\tan x$

(i) $y = 2^{-x}$

It is recommended that the graph for the original curve and its image be drawn on the same set of axes.

6. In this problem the points are rotated about the origin through an angle θ such that $\tan \theta = \frac{3}{4}$. In order to achieve the same result, we rotate the axes through an angle θ such that $\tan \theta = \frac{-3}{4}$. Thus $\sin \theta = -\frac{3}{5}$, $\cos \theta = \frac{4}{5}$, and the equations of rotation are

$$\begin{cases} x' = \frac{1}{5}(4x - 3y) \\ y' = \frac{1}{5}(3x + 4y) \end{cases} .$$

Under this rotation $A = (-2,1)$ maps into $A' = (-\frac{11}{5}, -\frac{2}{5})$, $B = (5,-2)$ maps into $B' = (\frac{26}{5}, \frac{7}{5})$, and $C = (3,3)$ maps into $C' = (\frac{3}{5}, \frac{21}{5})$.

Invariant properties are:

(a) Measure of Distance. For example, $d(A,B) = \sqrt{58} = d(A',B')$.

(b) Measure of Angle. For example,

$m_{\overline{AB}} = -\frac{3}{7}$, $m_{\overline{AC}} = \frac{2}{5}$; $\cos A = \frac{1}{\sqrt{2}}$, and $m\angle A = \frac{\pi}{4}$.

$m_{\overline{A'B'}} = \frac{9}{37}$, $m_{\overline{A'C'}} = \frac{23}{14}$, $\cos A' = \frac{1}{\sqrt{2}}$, and $m\angle A' = \frac{\pi}{4}$.

(c) Area of $\triangle ABC$ = Area of $\triangle A'B'C'$.

Apply the formula $s = \sqrt{s(s-a)(s-b)(s-c)}$ where s is the semi-perimeter.

7. We do not present the constructions here since the procedure is shown in the text. In part (b), each corresponding pair of lines must be mapped separately.

8. Since the points on the curves are rotated through an angle of measure $\frac{\pi}{6}$, the axes must be rotated through an angle of measure $-\frac{\pi}{6}$. The equations of rotation are, therefore:

$$\begin{cases} x = \frac{1}{2}(\sqrt{3}\,x' + y') \\ y = \frac{1}{2}(-x' + \sqrt{3}\,y') . \end{cases}$$

(a) The image of the line $3x + 2y - 8 = 0$ is
$\frac{3}{2}(\sqrt{3}\,x' + y') + \frac{2}{2}(-x' + \sqrt{3}\,y') - 8 = 0$, which simplifies to
$(3\sqrt{3} - 2)x' + (2\sqrt{3} + 3)y' - 16 = 0$.

(b) The image of the circle $x^2 + y^2 = 25$ is $x'^2 + y'^2 = 25$.

(c) The image of the parabola $y^2 = 4x$ is
$x'^2 - 2\sqrt{3}x'y' + 3y'^2 - 8\sqrt{3}x' - 8y' = 0$.

(NOTE: You may wish to excuse your students from sketching this parabola.)

9. Another way to write this transformation is

$$\begin{cases} x' = -y + 3 \\ y' = x + 1 \end{cases} \quad \text{or} \quad \begin{cases} x = y' - 1 \\ y = -x' - 3 . \end{cases}$$

The images of the curves in Exercise 8 are:

(a) The image of the line $3x + 2y - 8 = 0$ is
$3(y' - 1) + 2(-x' - 3) - 8 = 0$, which simplifies to
$2x' - 3y' + 17 = 0$. Note that these lines are perpendicular.

(b) The image of the circle $x^2 + y^2 = 25$ is the circle
$x'^2 + y'^2 + 6x' - 2y' - 15 = 0$ which has its center at $(-3,1)$ and a radius of 5.

(c) The image of the parabola $y^2 = 4x$ is $x'^2 + 6x' - 4y' + 13 = 0$.

10. Another way to write this transformation is:

$$\begin{cases} x' = x + y \\ y' = 2x - y \end{cases} \quad \text{or} \quad \begin{cases} x = \frac{1}{3}(x' + y') \\ y = \frac{1}{3}(2x' - y') \end{cases} .$$

The line $L_1 : 3x - 2y + 5 = 0$ maps into the line

$L_1' : x' - 5y' - 15 = 0$.

The line $L_2 : 3x - 2y - 3 = 0$ maps into the line

$L_2' : x' - 5y' + 9 = 0$.

$L_1' || L_2'$ since they have the same slope.

10-6. Inversions.

The justification for Section 6 was presented in Section 10-1 and in Section 10-5 of this commentary. This transformation has been studied by many outstanding mathematicians and plays a role in the Poincaré model of non-Euclidean geometry. You will find an excellent discussion of inversion geometry in Introduction to Higher Geometry by William C. Graustein.

An inversion is an involutory transformation as defined in Section 10-5 of this commentary. If an inversion T carries $P \longrightarrow P'$, a second application of T will carry $P' \longrightarrow P$.

You may like to point out to the class that as point P approaches the origin, the image P' will recede farther and farther out in the plane. For this reason it is often stated that the center of the circle of inversion corresponds to the "point at infinity" under the inversion. This is a useful concept since we can now say that an inversion sets up a one-to-one correspondence between the points of the plane and their images.

One of the most important properties of an inversion is that it transforms straight lines and circles into straight lines and circles. Specifically, we show that, after an inversion:

1. A line through the origin inverts into the same line through the origin, although the points on the line are interchanged.
2. A line not through the origin inverts into a circle through the origin.

3. A circle through the origin inverts into a straight line not through the origin.
4. A circle not through the origin inverts into a circle not through the origin.

You may want to precede Example 1 by a similar problem wherein the constants a , b , c are specified. You could then draw the unit circle of inversion, the straight line, and its inverse on the same set of coordinates. You can thus verify that the inverse really does pass through the origin.

This same comment holds for Examples 2 and 3 . In Example 2, let $x = \pm 1$, $x = \pm 2$, $y = \pm 1$, $y = \pm 2$ and observe what happens. It may also be profitable and interesting to explore with your class the inverses of a family of circles concentric to the unit circle; for example $x^2 + y^2 = 4$, $x^2 + y^2 = 9$, ... This could be followed by a set of concentric circles with their centers at $(2,4)$ or some other point.

If you have studied projective geometry or non-Euclidean geometry you undoubtedly recall the cross-ratio which appears in Exercise 9. The cross-ratio is invariant under a projective transformation and certain other transformations as well as under the inversion transformation. This property plays a very important role in the proof of the consistency of non-Euclidean geometry. If interested, you may like to read Chapter 4 of Foundations and Fundamental Concepts of Mathematics by Newsom and Eves.

Exercises 10-6

1. The inverse of the line $3x + 2y - 6 = 0$ is

$$\frac{3x'}{x'^2 + y'^2} + \frac{2y'}{x'^2 + y'^2} - 6 = 0 ,$$

which simplifies to $x'^2 + y'^2 - \frac{x'}{2} - \frac{y'}{3} = 0$. This represents a circle with center at $(\frac{1}{4}, \frac{1}{6})$ and radius $\frac{\sqrt{13}}{12}$. The circle passes through the origin.

2. The inverse of the line $y = 5x$ is the line $y' = 5x'$. The line inverts into itself.

3. The inverse of the line $y = 3$ is the curve $\frac{y'}{x'^2 + y'^2} = 3$, which simplifies to $x'^2 + y'^2 - \frac{1}{3}y' = 0$. It is a circle which has its center at $(0,\frac{1}{6})$ and radius $\frac{1}{6}$. The circle passes through the origin.

4. The inverse of the parabola $y^2 = 4x$ is the curve

$$\frac{y'^2}{(x'^2 + y'^2)^2} = \frac{4x'}{(x'^2 + y'^2)}, \text{ which simplifies to } y'^2 = \frac{4x'^3}{1 - 4x'} .$$

NOTE: The graph of this curve may be left as a challenge exercise. It is a cissoid with the following properties:

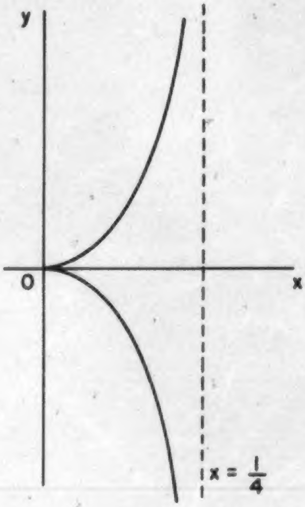
(1) Symmetry with respect to the x-axis.

(2) Intercept at $(0,0)$.

(3) Asymptote: $x' = \frac{1}{4}$.

(4) Extent: $0 < x' < \frac{1}{4}$.

The curve has this appearance



5. The inverse of the circle $(x - 4)^2 + (y - 4)^2 = 16$ is found as follows: The equation simplifies to $x^2 + y^2 - 8x - 8y + 16 = 0$. Apply the transformation and obtain:

$$\left(\frac{x'^2}{(x'^2 + y'^2)^2} + \frac{y'^2}{(x'^2 + y'^2)^2}\right) - \frac{8x'}{x'^2 + y'^2} - \frac{8y'}{x'^2 + y'^2} + 16 = 0$$

or $$\frac{1}{x'^2 + y'^2} - \frac{8x'}{x'^2 + y'^2} - \frac{8y'}{x'^2 + y'^2} + 16 = 0$$

or $1 - 8x' - 8y' + 16(x'^2 + y'^2) = 0$.

or $x'^2 + y'^2 - \frac{x'}{2} - \frac{y'}{2} + \frac{1}{16} = 0$ which represents a circle with center at $(\frac{1}{4}, \frac{1}{4})$ and radius $\frac{1}{4}$.

6. This problem is essentially solved in Example 2 of the text.

7. The inverse of the line $L : 3x + 2y - 6 = 0$ is the circle

$L' : x'^2 + y'^2 - \frac{x'}{2} - \frac{y'}{3} = 0$.

We now apply the inverse transformation

$$\begin{cases} x' = \dfrac{x}{x^2 + y^2} \\ y' = \dfrac{y}{x^2 + y^2} \end{cases} \quad \text{to } L' \text{ and obtain:}$$

$$\left(\frac{x^2}{(x^2 + y^2)^2} + \frac{y^2}{(x^2 + y^2)^2}\right) - \frac{x}{2(x^2 + y^2)} - \frac{y}{3(x^2 + y^2)} = 0 ,$$

or $$\frac{1}{x^2 + y^2} - \frac{x}{2(x^2 + y^2)} - \frac{y}{3(x^2 + y^2)} = 0 ,$$

or $3x + 2y - 6 = 0$, which we recognize as the original line L .

A reasonable conjecture is that a second application of the same inverse transformation yields the original curve. This verifies the fact that an inversion is an involutory transformation.

8. If we had not used a unit circle, we would have had $d(O,P) \cdot d(O,P') = r^2$

or $d(O,P) = \frac{r^2}{d(O,P')}$.

Since $\frac{d(O,P)}{d(O,P')} = \frac{x}{x'} = \frac{r^2}{(d(O,P'))^2} = r^2(d(O,P))^2$,

we have $x = \frac{r^2 x'}{x'^2 + y'^2}$ and $x' = \frac{r^2 x}{x^2 + y^2}$.

Likewise, $y = \frac{r^2 y'}{x'^2 + y'^2}$ and $y' = \frac{r^2 y}{x^2 + y^2}$.

9. Since $r = 2$, the inverse transformation is :

$$\begin{cases} x' = \dfrac{4x}{x^2 + y^2} \\ y' = \dfrac{4y}{x^2 + y^2} \end{cases}$$

The inverse points are as follows:

$$A = (0,-3) \longrightarrow A' = (0, -\tfrac{4}{3})$$

$$B = (1,-1) \longrightarrow B' = (2,-2)$$

$$C = (2,1) \longrightarrow C' = (\tfrac{8}{5}, \tfrac{4}{5})$$

$$D = (3,3) \longrightarrow D' = (\tfrac{2}{3}, \tfrac{2}{3})$$

$d(A,C) = 2\sqrt{5}$, $d(A,D) = 3\sqrt{5}$, $d(B,C) = \sqrt{5}$, $d(B,D) = 2\sqrt{5}$.

$$\frac{d(A,C)}{d(A,D)} \div \frac{d(B,C)}{d(B,D)} = \frac{2\sqrt{5}}{3\sqrt{5}} \div \frac{\sqrt{5}}{2\sqrt{5}} = \frac{4}{3} .$$

$d(A',C') = \frac{8}{3}$, $d(A',D') = \frac{2}{3}\sqrt{10}$, $d(B',C') = 2\sqrt{2}$, $d(B',D') = \frac{4}{3}\sqrt{5}$.

$$\frac{d(A',C')}{d(A',D')} \div \frac{d(B',C')}{d(B',D')} = \frac{\frac{8}{3}}{\frac{2}{3}\sqrt{10}} \div \frac{2\sqrt{2}}{\frac{4}{3}\sqrt{5}} = \frac{4}{3} .$$

The above verifies that an inversion is a cross-ratio preserving transformation.

10. The instructions for the construction are given in the text. As in the proof in the text for the first construction:

$$\angle ORP = \angle POR = \angle OPR' .$$

Thus $\triangle RP'O \sim \triangle ORP$ and

$$\frac{d(O,P')}{d(O,R)} = \frac{d(O,R)}{d(O,P)} \quad \text{or} \quad d(O,P) \cdot d(O,P') = r^2 .$$

Review Exercises

Before proceeding with the solutions of this set of problems, we feel it important to point out that there are two ways to interpret the mapping symbol, such as $(x,y) \longrightarrow (2x,3y)$, which appears in the first exercise.

In this text, we have adopted the convention that $(x,y) \longrightarrow (2x,3y)$ is merely another way of writing the transformation

$$\begin{cases} x' = 2x \\ y' = 3y , \end{cases} \quad \text{or} \quad \begin{cases} x = \dfrac{x'}{2} \\ y = \dfrac{y'}{3} . \end{cases}$$

A second interpretation, which is often used (but not in this text), is the following: wherever an x appears in the equation, replace it by $2x$; wherever a y appears, replace it by $3y$.

The first interpretation leads to the result $x'^2 = \frac{8}{3}y'$; the second interpretation leads to the result $x^2 = \frac{3}{2}y$ when applied to Exercise 1(a) .

Solutions

1. (a) $x'^2 = \frac{8}{3}y'$ (See above)

   (b) The transformation can be written as

$$\begin{cases} x' = x + 2 \\ y' = 3y \end{cases} \quad \text{or} \quad \begin{cases} x = x' - 2 \\ y = \dfrac{y'}{3} . \end{cases}$$

   Applying this transformation to the parabola $x^2 = 2y$, we have $(x' - 2)^2 = 2\frac{y'}{3}$, which simplifies to $3x'^2 - 12x' - 2y' + 12 = 0$. An invariant property of this transformation is that a parabola maps into a parabola; i.e., the type of curve is invariant.

   (c) The transformation can be written as

$$\begin{cases} x' = x - 1 \\ y' = y + 2 \end{cases} \quad \text{or} \quad \begin{cases} x = x' + 1 \\ y = y' - 2 . \end{cases}$$

   The parabola $x^2 = 2y$ transforms into the curve: $(x' + 1)^2 = 2(y' - 2)$ which we recognize as the equation of the same parabola with respect to a new origin at $(-1,2)$.

2. The mapping $(x,y) \longrightarrow (kx,ky)$ can be written as the transformation

$$\begin{cases} x' = kx \\ y' = ky \end{cases} \quad \text{or} \quad \begin{cases} x = \frac{x'}{k} \\ y = \frac{y'}{k} \end{cases}$$

In this case, we let $k = 2$. Applying this transformation, in turn, to each of the curves, we have:

(a) The line $2x + 3y - 6 = 0$ transforms to $2x' + 3y' - 12 = 0$, a line parallel to the original line.

(b) The circle $x^2 + y^2 = 25$ transforms to $x'^2 + y'^2 = 100$, a circle with the same center but with a radius twice as large.

(c) The parabola $y^2 = -4x$ transforms to the parabola $y'^2 = -8x$, which is "parallel" to the original curve.

The title is well justified since a figure "similar" to the original appears after the transformation.

3. Under this transformation, the image of L_1 is the line $L_1' : x - 5y - 8 = 0$, and the image of L_2 is the line $L_2' : 5x + y - 12 = 0$. $L_1 \perp L_2$ since $m_1' \cdot m_2' = -1$.

4. Applying the transformation T to each of the curves, we obtain the following images:

(a) $13x'^2 - 4x'y' + 20y'^2 - 20x' - 24y' + 13 = 0$. (An ellipse)

(b) $65x'^2 + 172x'y' - 28y'^2 + 232x' + 112y' - 176 = 0$. (A hyperbola)

(c) $x + 22y - 28 = 0$.

(d) $x + 22y - 17 = 0$.

Lines (c) and (d) are parallel and their images are parallel. A reasonable conjecture is that an affine transformation preserves the property of parallelism. (It does.)

5. The proof follows that given in the text for the mapping $(x,y) \longrightarrow (x,-y)$ in Section 10-5.